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A C^4-Spline Collocation Method for Systems of solving Higher Index Differential-Algebraic Equations

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Abstract: **In this paper, five-point spline collocation method is considered for solving differential-algebraic systems with index greater than or equal one. The study shows that the purposed method applied to higher index differential-algebraic equations is stable and the order of convergence is eight for index greater than or equals two, while the order is nine for index-1. Numerical experiments are presented that illustrate the theoretical results. The method is also compared with the method given by Mahmoud [7], Tabatabaei and Celik [11] and Dhamacharoen [13] . This comparison shows that the presented method is more accurate than the previous others.**

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1. INTRODUCTION

Differential algebraic equations (DAEs) arise in many instances when using mathematical modeling techniques for describing phenomena in science, engineering, economics, etc. In the last three decades, the use of differential algebraic equations has become standard modeling practice in many applications, such as constrained mechanics and chemical process simulations. In most cases, the model is too complex to allow one to find an exact solution or even an approximate solution by hand: an efficient, reliable computer simulation is required. It is well known that DAEs can be difficult to solve when they have a higher index, i.e., an index greater than one (cf. [**4**]). Higher-index DAEs are ill posed in a certain sense, especially when the index is greater than two [**2**], and a straightforward discretization generally does not work well. Some numerical methods have been developed, using Runge–Kutta, BDF and regularization methods [**3**,**5**,**8**,**10**]. Analytical approximate solutions of systems of differential-algebraic equations by Laplace homotopy analysis method in [**1**]. Quintic C^2- spline collocation methods for solving initial value problems in higher index differential-algebraic equations are presented in [**7**]. Tabatabaei and Celik have found the numerical solution of differential-algebraic equations with index-3 by Pade approximation in [**11**].

In this paper, we study the error analysis and order of convergence of ninth spline collocation method (NSCM) applied to solvable linear constant coefficient DAEs

$$\mathbf{A}\,y'+\mathbf{B}\,y=g(x)\,, \quad x\in[a,b] \tag{1}$$

of arbitrary index-v, where A and B are square constant matrices and $g(x)$ is a smooth function. Moreover, the strict stability properties of NSCM are discussed when it is applied to nonlinear systems of DAEs of the form,

$$F\,[\,x\,,\; y(x),\; y'(x)]=0\,, \quad x\in[a,b] \tag{2}$$

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where the initial values of $y(x_0)$ are given and F is linear in y'.

The paper is organized as follows: In Section 2, the case of linear constant coefficient index-v systems is studied. It shows that the 5-point spline collocation method when applied index-1 systems are stable, consistent of order **nine** , and convergent of order **nine**. After that, we generalize the NSCM when applied to differential-algebraic systems with index greater than one. It turns out that proposed NSCM are stable and consistent of order eight for all $v \geq 2$. In Section 3 the NSCM are shown to be strictly stable if applied to index-v DAEs. Numerical problems are provided in Section 4 to test the efficiency of the NSCM when applied to differential-algebraic systems.

2. DESCRIPTION NINTH SPLINE COLLOCATION METHOD

The purposed method uses five collocation points:

$$x_{i-1+z_j} = x_{i-1} + z_j h, \; j = 1(1)5 \tag{3}$$

where collocation parameters are given

$$0 < z_1 < z_2 < z_3 < z_4 < z_5 = 1$$

Denote by $x_i = a + i\,h,\; i = 0(1)N$, the grid points of the uniform partition of $[a,b]$ into subintervals $\mathrm{I}_i = [x_{i-1}, x_i],\; i = 1(1)N$, and $h = (b-a)/N$ is the constant stepsize. Ninth C^4-spline functions $S(x)$ can be represented on each I_i by

$$\begin{aligned} S(x) = {} & (1 - 126\gamma^5 + 420\gamma^6 - 540\gamma^7 + 315\gamma^8 - 70\gamma^9) S_{i-1}^{[0]} + (\gamma - 70\gamma^5 - 224\gamma^6 - 280\gamma^7 \\ & + 160\gamma^8 - 35\gamma^9) S_{i-1}^{[1]} + (\tfrac{1}{2}\gamma^2 - \tfrac{35}{2}\gamma^5 + \tfrac{105}{2}\gamma^6 - 63\gamma^7 + 35\gamma^8 - \tfrac{15}{2}\gamma^9) S_{i-1}^{[2]} + (\tfrac{1}{6}\gamma^3 - \tfrac{5}{2}\gamma^5 \\ & - \tfrac{20}{3}\gamma^6 - \tfrac{15}{2}\gamma^7 + 4\gamma^8 - \tfrac{5}{6}\gamma^9) S_{i-1}^{[3]} + (\tfrac{1}{24}\gamma^4 - \tfrac{5}{24}\gamma^5 + \tfrac{5}{12}\gamma^6 - \tfrac{5}{12}\gamma^7 + \tfrac{5}{24}\gamma^8 - \tfrac{1}{24}\gamma^9) S_{i-1}^{[4]} + \\ & (126\gamma^5 - 420\gamma^6 + 540\gamma^7 - 315\gamma^8 + 70\gamma^9) S_i^{[0]} + (-56\gamma^5 + 196\gamma^6 - 260\gamma^7 + 155\gamma^8 - 35\gamma^9) S_i^{[1]} + \\ & (\tfrac{21}{2}\gamma^5 - \tfrac{77}{2}\gamma^6 + 53\gamma^7 - \tfrac{65}{2}\gamma^8 + \tfrac{15}{2}\gamma^9) S_i^{[2]} + (-\gamma^5 + \tfrac{23}{6}\gamma^6 - \tfrac{11}{2}\gamma^7 - \tfrac{7}{2}\gamma^8 - \tfrac{5}{6}\gamma^9) S_i^{[3]} \\ & + (\tfrac{1}{24}\gamma^5 - \tfrac{1}{6}\gamma^6 + \tfrac{1}{4}\gamma^7 - \tfrac{1}{6}\gamma^8 + \tfrac{1}{24}\gamma^9) S_i^{[4]} \end{aligned} \tag{4}$$

where $x \in [x_{i-1}, x_i], \gamma = (x - x_{i-1})/h \in [0,1]$ and denotations:

$$\begin{aligned} & S_{i-1}^{[0]} = S(x_{i-1}), \quad S_{i-1}^{[1]} = hS'(x_{i-1}), \quad S_{i-1}^{[2]} = h^2 S''(x_{i-1}), \\ & S_{i-1}^{[3]} = h^3 S'''(x_{i-1}), \quad S_{i-1}^{[4]} = h^4 S^{(4)}(x_{i-1}), \qquad i = 1(1)N \end{aligned} \tag{5}$$

From equation (4), it follows that

$$\begin{aligned} hS'(x) = {} & (-630\gamma^4 + 2520\gamma^5 - 3780\gamma^6 + 2520\gamma^7 - 630\gamma^8) S_{i-1}^{[0]} + (1 - 350\gamma^4 + 1344\gamma^5 \\ & - 1960\gamma^6 + 1280\gamma^7 - 315\gamma^8) S_{i-1}^{[1]} + (\gamma - \tfrac{175}{2}\gamma^4 + 315\gamma^5 - 441\gamma^6 + 280\gamma^7 - \tfrac{135}{2}\gamma^8) S_{i-1}^{[2]} + \\ & (\tfrac{1}{2}\gamma^2 - \tfrac{25}{2}\gamma^4 + 40\gamma^5 - \tfrac{105}{2}\gamma^6 + 32\gamma^7 - \tfrac{15}{2}\gamma^8) S_{i-1}^{[3]} + (\tfrac{1}{6}\gamma^3 - \tfrac{25}{24}\gamma^4 + \tfrac{5}{2}\gamma^5 - \tfrac{35}{12}\gamma^6 + \tfrac{5}{3}\gamma^7 \\ & - \tfrac{7}{8}\gamma^8) S_{i-1}^{[4]} + (630\gamma^4 - 2520\gamma^5 + 3780\gamma^6 - 2520\gamma^7 + 630\gamma^8) S_i^{[0]} + (-280\gamma^4 + 1176\gamma^5 - 1820\gamma^6 \\ & + 1240\gamma^7 - 315\gamma^8) S_i^{[1]} + (\tfrac{105}{2}\gamma^4 - 231\gamma^5 + 371\gamma^6 - 260\gamma^7 + \tfrac{135}{2}\gamma^8) S_i^{[2]} + \\ & (-5\gamma^4 + 23\gamma^5 - \tfrac{77}{2}\gamma^6 + 28\gamma^7 - \tfrac{15}{2}\gamma^8) S_i^{[3]} + (\tfrac{5}{24}\gamma^4 - \gamma^5 + \tfrac{7}{4}\gamma^6 - \tfrac{4}{3}\gamma^7 + \tfrac{3}{8}\gamma^8) S_i^{[4]} \end{aligned} \tag{6}$$

The spline approximations (4)-(6) are applied into (1) such that in each subinterval $I_i = [x_{i-1}, x_i]$, the following five collocation conditions

$$\mathbf{A}\, S'(x_{i-1+z_j}) + \mathbf{B}\, S(x_{i-1+z_j}) = g(x_{i-1+z_j}),\ j = (1)5 \quad , \quad i = 1(1)\mathrm{N} \tag{7a}$$

hold, where initial-values:

$$\begin{aligned} & S(x_0) = y_0,\ S'(x_0) = y_0',\ S''(x_0) = y_0'', \\ & S'''(x_0) = y_0''' , \quad S^{(4)}(x_0) = y_0^{(4)} \end{aligned} \tag{7b}$$

The differential and algebraic parts of the system (7a) can be completely decoupled from each other [4]. Thus, it is sufficient to study the differential and algebraic parts separately

to get an understanding of the general linear constant-coefficient DAE.

Consider then a canonical algebraic subsystem of index-v

$$M\, y' + y = g(x) \tag{8}$$

where M is a $v \times v$ matrix, $g(x) = (g_1(x), \cdots, g_v(x))^T$, and $y(x) = (y_1(x), \cdots, y_v(x))^T$.

The solution to (2.5) is given by

$$\begin{aligned} y_1(x) &= g_1(x) \\ y_2(x) &= g_2(x) - y_1'(x) \\ &\vdots \\ y_v(x) &= g_v(x) + \sum_{i=1}^{v-1} (-1)^{v-i}\, y_i^{(v-i)}(x). \end{aligned}$$

Note that the j th component exhibits the index j behavior of the system, in the sense that $y_j(x)$ depends on the (j-1)st derivative of the input function $g(x)$. Applying the method to (8), we obtain

$$M\, S'(x_{i-1+z_j}) + S(x_{i-1+z_j}) = g(x_{i-1+z_j}) \ ,\quad j = 1(1)5 \ ,\quad i = 1(1)N \tag{9}$$

Let $S = (S_1, \ldots, S_v)^T$, $S' = (S_1', \ldots, S_v')^T$, where S'_j denotes the derivative corresponding to the j th component of the solution vector.

2.1-Ninth Spline Method for Indix-1 DAE:

First, we assume that the method is applied to index-1 system, then (9) reduces to a set of algebraic equations of the form

$$S_1(x_{i-1+z_j}) = g_1(x_{i-1+z_j}) \ ,\quad j = 1(1)5 \ ,\quad i = 1(1)N \tag{10}$$

Using the approximation (4) into (10) , i.e., taking $S_1 \equiv S$, we have

$$\begin{aligned} &(126z_j^5 - 420z_j^6 + 540z_j^7 - 315z_j^8 + 70z_j^9)S_{1,i}^{[0]} + (-56z_j^5 + 196z_j^6 - 260z_j^7 + 155z_j^8 - 35z_j^9)S_{1,i}^{[1]} + \\ &(\tfrac{21}{2}z_j^5 - \tfrac{77}{2}z_j^6 + 53z_j^7 - \tfrac{65}{2}z_j^8 + \tfrac{15}{2}z_j^9)S_{1,i}^{[2]} + (-z_j^5 + \tfrac{23}{6}z_j^6 - \tfrac{11}{2}z_j^7 - \tfrac{7}{2}z_j^8 - \tfrac{5}{6}z_j^9)S_{1,i}^{[3]} \\ &+(\tfrac{1}{24}z_j^5 - \tfrac{1}{6}z_j^6 + \tfrac{1}{4}z_j^7 - \tfrac{1}{6}z_j^8 + \tfrac{1}{24}z_j^9)S_{1,i}^{[4]} = -(1 - 126z_j^5 + 420z_j^6 - 540z_j^7 + 315z_j^8 - 70z_j^9)S_{1,i-1}^{[0]} \\ &-(z_j - 70z_j^5 - 224z_j^6 - 280z_j^7 + 160z_j^8 - 35z_j^9)S_{1,i-1}^{[1]} - (\tfrac{1}{2}z_j^2 - \tfrac{35}{2}z_j^5 + \tfrac{105}{2}z_j^6 - 63z_j^7 + 35z_j^8 \\ &- \tfrac{15}{2}z_j^9)S_{1,i-1}^{[2]} - (\tfrac{1}{6}z_j^3 - \tfrac{5}{2}z_j^5 - \tfrac{20}{3}z_j^6 - \tfrac{15}{2}z_j^7 + 4z_j^8 - \tfrac{5}{6}z_j^9)S_{1,i-1}^{[3]} - (\tfrac{1}{24}z_j^4 - \tfrac{5}{24}z_j^5 + \tfrac{5}{12}z_j^6 - \tfrac{5}{12}z_j^7 \\ &+ \tfrac{5}{24}z_j^8 - \tfrac{1}{24}z_j^9)S_{1,i-1}^{[4]} + g_1(x_{i-1+z_j}), \quad j = 1(1)5 \end{aligned} \tag{11.a}$$

$$S_{1,i}^{[0]} = g_1(x_i) \ , \tag{11.b}$$

Substituting $S_{1,i}^{[0]} = g_1(x_i),\ S_{1,i-1}^{[0]} = g_1(x_{i-1})$ into (11.a), we have:

$$\begin{aligned}
&(-56z_j^5 + 196z_j^6 - 260z_j^7 + 155z_j^8 - 35z_j^9)S_{1,i}^{[1]} + (\tfrac{21}{2}z_j^5 - \tfrac{77}{2}z_j^6 + 53z_j^7 - \tfrac{65}{2}z_j^8 + \tfrac{15}{2}z_j^9)S_{1,i}^{[2]} \\
&+ (-z_j^5 + \tfrac{23}{6}z_j^6 - \tfrac{11}{2}z_j^7 - \tfrac{7}{2}z_j^8 - \tfrac{5}{6}z_j^9)S_{1,i}^{[3]} + (\tfrac{1}{24}z_j^5 - \tfrac{1}{6}z_j^6 + \tfrac{1}{4}z_j^7 - \tfrac{1}{6}z_j^8 + \tfrac{1}{24}z_j^9)S_{1,i}^{[4]} = \\
&-(z_j - 70z_j^5 - 224z_j^6 - 280z_j^7 + 160z_j^8 - 35z_j^9)S_{1,i-1}^{[1]} - (\tfrac{1}{2}z_j^2 - \tfrac{35}{2}z_j^5 + \tfrac{105}{2}z_j^6 - 63z_j^7 + 35z_j^8 \\
&- \tfrac{15}{2}z_j^9)S_{1,i-1}^{[2]} - (\tfrac{1}{6}z_j^3 - \tfrac{5}{2}z_j^5 - \tfrac{20}{3}z_j^6 - \tfrac{15}{2}z_j^7 + 4z_j^8 - \tfrac{5}{6}z_j^9)S_{1,i-1}^{[3]} - (\tfrac{1}{24}z_j^4 - \tfrac{5}{24}z_j^5 + \tfrac{5}{12}z_j^6 \\
&- \tfrac{5}{12}z_j^7 + \tfrac{5}{24}z_j^8 - \tfrac{1}{24}z_j^9)S_{1,i-1}^{[4]} - (1 - 126z_j^5 + 420z_j^6 - 540z_j^7 + 315z_j^8 - 70z_j^9)g_1(x_{i-1}) \\
&+ g_1(x_{i-1+z_j}) - (126z_j^5 - 420z_j^6 + 540z_j^7 - 315z_j^8 + 70z_j^9)g_1(x_i), \quad j = 1(1)5
\end{aligned} \tag{12.a}$$

Which are equivalent to the following recurrence formula:

$$\mathrm{A}_1 \underline{S}_{1,\,i} = \mathrm{B}_1 \underline{S}_{1,\,i-1} + \mathrm{D}_1 \underline{g}_{1,\,i} \ , i = 1(1)\mathrm{N} \tag{12.b}$$

where

$\mathrm{A}_1 = (a_{k,j}^1), \mathrm{B}_1 = (b_{k,j}^1)$ and $\mathrm{D}_1 = (d_{k,l}^1)$ are defined by

$$a_{k,1}^1 = -56z_j^5 + 196z_j^6 - 260z_j^7 + 155z_j^8 - 35z_j^9,\ a_{k,2}^1 = \tfrac{21}{2}z_j^5 - \tfrac{77}{2}z_j^6 + 53z_j^7 - \tfrac{65}{2}z_j^8 + \tfrac{15}{2}z_j^9$$

$$a_{k,3}^1 = -z_j^5 + \tfrac{23}{6}z_j^6 - \tfrac{11}{2}z_j^7 - \tfrac{7}{2}z_j^8 - \tfrac{5}{6}z_j^9,\ \ a_{k,4}^1 = \tfrac{1}{24}z_j^5 - \tfrac{1}{6}z_j^6 + \tfrac{1}{4}z_j^7 - \tfrac{1}{6}z_j^8 + \tfrac{1}{24}z_j^9,$$

$$b_{k,1}^1 = -(z_j - 70z_j^5 - 224z_j^6 - 280z_j^7 + 160z_j^8 - 35z_j^9),\ b_{k,2}^1 = -(\tfrac{1}{2}z_j^2 - \tfrac{35}{2}z_j^5 + \tfrac{105}{2}z_j^6 - 63z_j^7 + 35z_j^8 - \tfrac{15}{2}z_j^9),$$

$$b_{k,3}^1 = -(\tfrac{1}{6}z_j^3 - \tfrac{5}{2}z_j^5 - \tfrac{20}{3}z_j^6 - \tfrac{15}{2}z_j^7 + 4z_j^8 - \tfrac{5}{6}z_j^9), b_{k,4}^1 = -(\tfrac{1}{24}z_j^4 - \tfrac{5}{24}z_j^5 + \tfrac{5}{12}z_j^6 - \tfrac{5}{12}z_j^7 + \tfrac{5}{24}z_j^8 - \tfrac{1}{24}z_j^9),$$

$$d_{k,1}^1 = -(1 - 126z_j^5 + 420z_j^6 - 540z_j^7 + 315z_j^8 - 70z_j^9),$$

$$d_{k,6}^1 = -(126z_j^5 - 420z_j^6 + 540z_j^7 - 315z_j^8 + 70z_j^9),\ d_{k,k+1}^1 = 1,\ k=1(1)4,$$

$$d_{13}^1 = d_{14}^1 = d_{15}^1 = d_{22}^1 = d_{24}^1 = d_{25}^1 = d_{32}^1 = d_{33}^1 = d_{35}^1 = d_{42}^1 = d_{43}^1 = d_{44}^1 = 0,$$

and

$$\underline{S}_{1,\,i} = (S_{1,\,i}^{(1)}, S_{1,\,i}^{(2)}, S_{1,\,i}^{(3)}, S_{1,\,i}^{(4)})^T,\ \underline{S}_{1,\,i-1} = (S_{1,\,i-1}^{(1)}, S_{1,\,i-1}^{(2)}, S_{1,\,i-1}^{(3)}, S_{1,\,i-1}^{(4)})^T,$$

$$\underline{g}_{1,\,i} = (g_{1,\,i-1}, g_{1,\,i-1+z_1}, g_{1,\,i-1+z_2}, g_{1,\,i-1+z_3}, g_{1,\,i-1+z_4}, g_{1,\,i})^T.$$

Multiplying (12.a) by A_1^{-1}, we get

$$\underline{S}_{1,\,i} = \tilde{\mathrm{A}}_1 \underline{S}_{1,\,i-1} + \mathrm{A}_1^{-1}\mathrm{D}_1 \underline{g}_{1,\,i} \quad ,i = 1(1)\mathrm{N}, \tag{13}$$

where

$$\tilde{\mathrm{A}}_1 = \mathrm{A}_1^{-1}\mathrm{B}_1 \tag{14}$$

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If $0 < z_1 < z_2 < z_3 < z_4 < 1$, then $\tilde{A}_1$ exists because

$$|\tilde{A}_1| = \frac{(1-z_1)^4(1-z_2)^4(1-z_3)^4(1-z_4)^4}{z_1^4 z_2^4 z_3^4 z_4^4} \neq 0.$$

Definition 1 [6]: The NSCM (13) is called stable if $\|(\tilde{A}_1)^n\| \le \kappa = const$ for all $n \ge 1$, where $\kappa = \max_{1 \le i \le 4} \sum_{j=1}^{4} |a_{i,j}^n|$, $\tilde{A}_1^n = (a_{ij}^n)$, and $\tilde{A}_1$ is the matrix (14).

Theorem 1. The NSCM applied to index-1 systems is stable if eigenvalues of the matrix $\tilde{A}_1$ satisfy:

$$|\lambda_j| \le 1, \ j = 1(1)4 \qquad (15)$$

Proof. Since $\tilde{A}_1$ has four different eigenvalues, we can easily compute the eigenvalues of the matrix $\tilde{A}_1$ and show in Table1 that these values satisfy (15) for some z_1, z_2, z_3, z_4. Thus, according to the definition 1, the spline method (13) is stable because $\|\tilde{A}_1^n\|_\infty \le \kappa$, for $n \ge 1$, is uniformly bounded where $\kappa = \max_{1 \le i \le 4} \sum_{j=1}^{4} |a_{ij}^n|$, $\tilde{A}_1^n = (a_{ij}^n)$. In Table2, we show that $\kappa \to 0$ as $n \to \infty$, for the method with collocation parameters

($z_1 = 0.75, z_2 = 0.85, z_3 = 0.96, z_4 = 0.999$).

Table 1: Some stability intervals, which satisfy (2.12)

$0.55 \le z_1$	$0.99 < z_2 < z_3 < z_4 < 1$
$0.60 \le z_1$	$0.96 < z_2 < z_3 < z_4 < 1$
$0.65 \le z_1$	$0.93 < z_2 < z_3 < z_4 < 1$
$0.70 \le z_1$	$0.90 < z_2 < z_3 < z_4 < 1$
$0.75 \le z_1$	$0.87 < z_2 < z_3 < z_4 < 1$
$0.80 \le z_1$	$0.84 < z_2 < z_3 < z_4 < 1$
$0.820 \le z_1$	$0.825 < z_2 < z_3 < z_4 < 1$
$z_1 = 0.75, z_2 = 0.85, z_3 = 0.96, z_4 = 0.999$	

Table 2: The norm $\|\tilde{A}_1^n\|_\infty$ is uniformly bounded for all $n \ge 1$

n	1	2	5	10	20	$n \to \infty$
$\|\tilde{A}_1^n\|_\infty$	k=656.999	k=323.887	k=38.338	k=1.094	k=8.9083E-4	$k \to 0$

Thus, the method is stable within these intervals.

To find the local error, let $y_1(x_{i-1}) = g_1(x_{i-1})$.

Theorem 2. The 5-point spline collocation method is consistent and is of order nine for linear constant-coefficient index-1 systems, for all z_1, z_2, z_3, z_4 shown in Table1.

Proof. The local discretization error of (13) at x_i is defined to be

$$\underline{d}_{i,1}=\begin{bmatrix} h\,y_1'(x_i)\\ h^2y_1''(x_i)\\ h^3y_1'''(x_i)\\ h^4y_1^{(4)}(x_i)\end{bmatrix}-\mathrm{A}_1^{-1}\mathrm{B}_1\begin{bmatrix} h\,y_1'(x_{i-1})\\ h^2y_1''(x_{i-1})\\ h^3y_1'''(x_{i-1})\\ h^4y_1^{(4)}(x_{i-1})\end{bmatrix}-\mathrm{A}_1^{-1}\mathrm{D}_1\begin{bmatrix} p_i(x_{i-1})\\ p_i(x_{i-1+z_1})\\ p_i(x_{i-1+z_2})\\ p_i(x_{i-1+z_3})\\ p_i(x_{i-1+z_4})\\ p_i(x_i)\end{bmatrix},\ i=1(1)\mathrm{N},$$

where

$$\begin{aligned}P_i(x)&=(1-126\gamma^5+420\gamma^6-540\gamma^7+315\gamma^8-70\gamma^9)y_{i-1}+(\gamma-70\gamma^5-224\gamma^6-280\gamma^7\\&+160\gamma^8-35\gamma^9)hy_{i-1}'+(\tfrac{1}{2}\gamma^2-\tfrac{35}{2}\gamma^5+\tfrac{105}{2}\gamma^6-63\gamma^7+35\gamma^8-\tfrac{15}{2}\gamma^9)h^2y_{i-1}''+(\tfrac{1}{6}\gamma^3-\tfrac{5}{2}\gamma^5\\&-\tfrac{20}{3}\gamma^6-\tfrac{15}{2}\gamma^7+4\gamma^8-\tfrac{5}{6}\gamma^9)h^3y_{i-1}'''+(\tfrac{1}{24}\gamma^4-\tfrac{5}{24}\gamma^5+\tfrac{5}{12}\gamma^6-\tfrac{5}{12}\gamma^7+\tfrac{5}{24}\gamma^8-\tfrac{1}{24}\gamma^9)h^4S_{i-1}^{(4)}+\\&(126\gamma^5-420\gamma^6+540\gamma^7-315\gamma^8+70\gamma^9)y_i+(-56\gamma^5+196\gamma^6-260\gamma^7+155\gamma^8-35\gamma^9)hy_i'+\\&(\tfrac{21}{2}\gamma^5-\tfrac{77}{2}\gamma^6+53\gamma^7-\tfrac{65}{2}\gamma^8+\tfrac{15}{2}\gamma^9)h^2y_i''+(-\gamma^5+\tfrac{23}{6}\gamma^6-\tfrac{11}{2}\gamma^7-\tfrac{7}{2}\gamma^8-\tfrac{5}{6}\gamma^9)h^3y_i'''\\&+(\tfrac{1}{24}\gamma^5-\tfrac{1}{6}\gamma^6+\tfrac{1}{4}\gamma^7-\tfrac{1}{6}\gamma^8+\tfrac{1}{24}\gamma^9)h^4S_i^{(4)}\end{aligned}$$

is the ninth Hermite interpolation polynomial which interpolates $y_1, y_1',\ldots, y_1^{(4)}$

at $x = x_{i-1}$ and $x = x_i$. Since $|p_i(x)-y_1(x)|\le Ch^{10}, x\in \mathrm{I}_i$ it follows that

$$\underline{d}_{i,1}=\underline{\tilde{d}}_{i,1}+O(h^{10}),\ \ i=1(1)\mathrm{N}$$

where

$$\underline{\tilde{d}}_{i,1}=\begin{bmatrix} h\,y_1'(x_i)\\ h^2y_1''(x_i)\\ h^3y_1'''(x_i)\\ h^4y_1^{(4)}(x_i)\end{bmatrix}-\mathrm{A}_1^{-1}\mathrm{B}_1\begin{bmatrix} h\,y_1'(x_{i-1})\\ h^2y_1''(x_{i-1})\\ h^3y_1'''(x_{i-1})\\ h^4y_1^{(4)}(x_{i-1})\end{bmatrix}-\mathrm{A}_1^{-1}\mathrm{D}_1\begin{bmatrix} y_1(x_{i-1})\\ y_1(x_{i-1+z_1})\\ y_1(x_{i-1+z_2})\\ y_1(x_{i-1+z_3})\\ y_1(x_{i-1+z_4})\\ y_1(x_i)\end{bmatrix}\qquad (16)$$

Now using Taylor's expansion

$$y_1(x)=q_9(x)+O(h^{10}), x\in[x_{i-1},x_i],\ \ y_1\in C^{10}[a,b]$$

and observing that the method is exact for polynomials of degree ≤ 9 (that means for $y_1 \equiv q_9$ we have $\underline{\tilde{d}}_{i,1}=\underline{d}_{i,1}=0$) we deduce, according to lemma 8.11 [6], that the method

is thus consistent and is of order nine for all z_1, z_2, z_3, z_4 satisfying Table1.

2.2 Ninth Spline Method for Indix-2 DAE:

Now, applying spline the method to index-2, relation (9) reduces to a set of the following algebraic equations:

$$S_1(x_{i-1+z_j}) = g_1(x_{i-1+z_j}), \quad (17)$$

$$S_2(x_{i-1+z_j}) = g_2(x_{i-1+z_j}) - S_1'(x_{i-1+z_j}) \ , \ j = 1(1)4 \ , \ i = 1(1)\mathrm{N} \quad (18)$$

Also, applying the approximations (4)-(6) to (18) we have

$$(126z_j^5 - 420z_j^6 + 540z_j^7 - 315z_j^8 + 70z_j^9)S_{2,i}^{[0]} + (-56z_j^5 + 196z_j^6 - 260z_j^7 + 155z_j^8 - 35z_j^9)S_{2,i}^{[1]} + (\tfrac{21}{2}z_j^5 - \tfrac{77}{2}z_j^6 + 53z_j^7 - \tfrac{65}{2}z_j^8 + \tfrac{15}{2}z_j^9)S_{2,i}^{[2]} + (-z_j^5 + \tfrac{23}{6}z_j^6 - \tfrac{11}{2}z_j^7 - \tfrac{7}{2}z_j^8 - \tfrac{5}{6}z_j^9)S_{2,i}^{[3]} + [(\tfrac{1}{24}z_j^5 - \tfrac{1}{6}z_j^6 + \tfrac{1}{4}z_j^7 - \tfrac{1}{6}z_j^8 + \tfrac{1}{24}z_j^9)S_{2,i}^{[4]} + (630z_j^4 - 2520z_j^5 + 3780z_j^6 - 2520z_j^7 + 630z_j^8)S_{1,i}^{[0]} + (-280z_j^4 + 1176z_j^5 - 1820z_j^6 + 1240z_j^7 - 315z_j^8)/hS_{1,i}^{[1]} + (\tfrac{105}{2}z_j^4 - 231z_j^5 + 371z_j^6 - 260z_j^7 + \tfrac{135}{2}z_j^8)S_{1,i}^{[2]} + (-5z_j^4 + 23z_j^5 - \tfrac{77}{2}z_j^6 + 28z_j^7 - \tfrac{15}{2}z_j^8)S_{1,i}^{[3]} + (\tfrac{5}{24}z_j^4 - z_j^5 + \tfrac{7}{4}z_j^6 - \tfrac{4}{3}z_j^7 + \tfrac{3}{8}z_j^8)S_{1,i}^{[4]}]/h =$$

$$-(1 - 126z_j^5 + 420z_j^6 - 540z_j^7 + 315z_j^8 - 70z_j^9)S_{2,i-1}^{[0]} - (z_j - 70z_j^5 - 224z_j^6 - 280z_j^7 + 160z_j^8 - 35z_j^9)S_{2,i-1}^{[1]} - (\tfrac{1}{2}z_j^2 - \tfrac{35}{2}z_j^5 + \tfrac{105}{2}z_j^6 - 63z_j^7 + 35z_j^8 - \tfrac{15}{2}z_j^9)S_{2,i-1}^{[2]} - (\tfrac{1}{6}z_j^3 - \tfrac{5}{2}z_j^5 - \tfrac{20}{3}z_j^6 - \tfrac{15}{2}z_j^7 + 4z_j^8 - \tfrac{5}{6}z_j^9)S_{2,i-1}^{[3]} - (\tfrac{1}{24}z_j^4 - \tfrac{5}{24}z_j^5 + \tfrac{5}{12}z_j^6 - \tfrac{5}{12}z_j^7 + \tfrac{5}{24}z_j^8 - \tfrac{1}{24}z_j^9)S_{2,i-1}^{[4]} - [(-630z_j^4 + 2520z_j^5 - 3780z_j^6 + 2520z_j^7 - 630z_j^8)S_{1,i-1}^{[0]} + (1 - 350z_j^4 + 1344z_j^5 - 1960z_j^6 + 1280z_j^7 - 315z_j^8)S_{1,i-1}^{[1]} + (z_j - \tfrac{175}{2}z_j^4 + 315z_j^5 - 441z_j^6 + 280z_j^7 - \tfrac{135}{2}z_j^8)S_{1,i-1}^{[2]} + (\tfrac{1}{2}z_j^2 - \tfrac{25}{2}z_j^4 + 40z_j^5 - \tfrac{105}{2}z_j^6 + 32z_j^7 - \tfrac{15}{2}z_j^8)S_{1,i-1}^{[3]} + (\tfrac{1}{6}z_j^3 - \tfrac{25}{24}z_j^4 + \tfrac{5}{2}z_j^5 - \tfrac{35}{12}z_j^6 + \tfrac{5}{3}z_j^7 - \tfrac{7}{8}z_j^8)S_{1,i-1}^{[4]}]/h + g_2(x_{i-1+z_j}), \quad j = 1(1)5 \quad (19)$$

$$S_{2,i}^{(0)} = g_2(x_i) - S_{1,\,i}^{(1)}/h \ , \ S_{2,\,i-1}^{(0)} = g_2(x_{i-1}) - S_{1,\,i-1}^{(1)}/h \ . \quad (20)$$

Substituting (20) into (19), we get the following recurrence formula:

$$\mathrm{A}_2\,\underline{S}_{2,\,i} = \mathrm{B}_2\,\underline{S}_{2,\,i-1} + \mathrm{D}_2\,\underline{g}_{2,\,i}, \ i = 1(1)\mathrm{N}, \quad (21)$$

where

$$\mathrm{A}_2 = (a_{i,j}^2), \mathrm{B}_2 = (b_{i,j}^2) \text{ and } \mathrm{D}_2 = (d_{i,k}^2),$$

$$\underline{S}_{2,\,i} = (S_{1,\,i}^{(1)}, S_{1,\,i}^{(2)}, S_{1,\,i}^{(3)}, S_{1,\,i}^{(4)}, S_{2,\,i}^{(1)}, S_{2,\,i}^{(2)}, S_{2,\,i}^{(3)}, S_{2,\,i}^{(4)})^T,$$

$$\underline{S}_{2,\,i-1} = (S_{1,\,i-1}^{(1)}, S_{1,\,i-1}^{(2)}, S_{1,\,i-1}^{(3)}, S_{1,\,i-1}^{(4)}, S_{2,\,i-1}^{(1)}, S_{2,\,i-1}^{(2)}, S_{2,\,i-1}^{(3)}, S_{1,\,i-1}^{(4)})^T,$$

$$\underline{g}_{2,\,i} = (g_{1,\,i-1}, g_{1,\,i-1+z_1}, g_{1,\,i-1+z_2}, g_{1,\,i-1+z_3}, g_{1,\,i-1+z_4}, g_{1,\,i}, g_{2,\,i-1}, g_{2,\,i-1+z_1}, g_{2,\,i-1+z_2}, g_{2,\,i-1+z_3}, g_{2,\,i-1+z_4}, g_{2,\,i})^T.$$

Multiplying (2.19) by A_2^{-1}, we get

$$\underline{S}_{2,\,i} = \widetilde{\mathrm{A}}_2\,\underline{S}_{2,\,i-1} + \mathrm{A}_2^{-1}\mathrm{D}_2\,\underline{g}_{2,\,i}, \quad (22)$$

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where $\tilde{A}_2 = A_2^{-1} B_2$.

We can easily find that $\tilde{A}_2$ has the form

$$\tilde{A}_2 = \left[\begin{array}{c:c} \tilde{A}_1 & 0 \\ \hdashline 0 & \tilde{A}_1 \end{array}\right], \text{ where } \tilde{A}_1 \text{ is the matrix (14).}$$

2.3 Ninth Spline Method for Indix-*v* DAE:

In general, suppose that the method is applied to index-*v*, then (9) becomes:

$$S_1(x_{i-1+z_j}) = g_1(x_{i-1+z_j}), \tag{23a}$$

$$S_2(x_{i-1+z_j}) = g_2(x_{i-1+z_j}) - S_1'(x_{i-1+z_j}), \tag{23b}$$

$$\vdots$$

$$S_\nu(x_{i-1+z_j}) = g_\nu(x_{i-1+z_j}) + \sum_{r=1}^{\nu-1} (-1)^{\nu-r} S_r^{(\nu-r)}(x_{i-1+z_j}),\ j = 1(1)4,\ i = 1(1)N \tag{23c}$$

Using the approximations (4)-(6) to (23a)-(23c) we get

$$\underline{S}_{\nu,\,i} = \tilde{A}_\nu\, \underline{S}_{\nu,\,i-1} + A_\nu^{-1} D_\nu\, \underline{g}_{\nu,\,i}, \tag{24}$$

where to this end, $\tilde{A}_\nu$ can be found after tedious calculations, as

$$\tilde{A}_\nu = \left[\begin{array}{c:c:c:c} \tilde{A}_1 & 0 & \cdots & 0 \\ \hdashline 0 & \tilde{A}_1 & \ddots & \vdots \\ \hdashline \vdots & \ddots & \ddots & 0 \\ \hdashline 0 & \cdots & 0 & \tilde{A}_1 \end{array}\right]$$

where $\tilde{A}_\nu$ is a $4\nu \times 4\nu$ matrix, and which yields the following corollary.

Corollary 1. The 5-point spline collocation method is stable for linear constant-coefficient index-ν DAEs if.

$$|\lambda_j| \le 1,\ j = 1(1)4,$$

where $\lambda_j,\ j = 1(1)4$ are the eigenvalues of the matrix $\tilde{A}_1$ the defined by (23).

Proof. Note first that If $0 < z_1 < z_2 < z_3 < z_4 < 1$, then $\tilde{A}_\nu$ is existed because $|\tilde{A}_\nu| = \dfrac{(1-z_1)^{4\nu}(1-z_2)^{4\nu}(1-z_3)^{4\nu}(1-z_4)^{4\nu}}{z_1^{4\nu}\, z_2^{4\nu}\, z_3^{4\nu}\, z_4^{4\nu}} \ne 0$. Since $\tilde{A}_\nu$ has the same eigenvalues of $\tilde{A}_1$ with multiplicity ν, then according to Theorem 1 we find that four eigenvalues satisfy (15) for the same z_1, z_2, z_3, z_4 listed in Table1. □

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Finally, for algebraic subsystem of index-v, the local error satisfies

$$\underset{\sim}{\tilde{d}}_{i,v}=\begin{bmatrix} h\,y_1'(x_i)\\ h^2y_1''(x_i)\\ h^3y_1'''(x_i)\\ h^4y_1^{(4)}(x_i)\\ \vdots\\ h\,y_v'(x_i)\\ h^2y_v''(x_i)\\ h^3y_v'''(x_i)\\ h^4y_v^{(4)}(x_i)\end{bmatrix}-\mathrm{A}_v^{-1}\mathrm{B}_v\begin{bmatrix} h\,y_1'(x_{i-1})\\ h^2y_1''(x_{i-1})\\ h^3y_1'''(x_{i-1})\\ h^4y_1^{(4)}(x_{i-1})\\ \vdots\\ h\,y_v'(x_{i-1})\\ h^2y_v''(x_{i-1})\\ h^3y_v'''(x_{i-1})\\ h^4y_v^{(4)}(x_{i-1})\end{bmatrix}-\mathrm{A}_v^{-1}\mathrm{D}_v\begin{bmatrix} g_1(x_{i-1})\\ g_1(x_{i-1+z_1})\\ g_1(x_{i-1+z_2})\\ g_1(x_{i-1+z_3})\\ g_1(x_{i-1+z_4})\\ g_1(x_i)\\ \vdots\\ g_v(x_{i-1})\\ g_v(x_{i-1+z_1})\\ g_v(x_{i-1+z_2})\\ g_v(x_{i-1+z_3})\\ g_v(x_{i-1+z_4})\\ g_v(x_i)\end{bmatrix}$$

Using Taylor's expansion

$$y_j(x)=q_{9,j}(x)+O(h^{10}),x\in[x_{i-1},x_i],\;\; y_j\in C^{10}[a,b],\; j=1(1)v$$

We get

$$\begin{aligned} y_v(x) &= g_v(x)-\sum_{i=1}^{v-1}(-1)^{v-i}y_i^{(v-i)}(x)=g_v(x)+(-1)^{v-1}y_{v-1}'(x)\\ &=\sum_{r=0}^{9}\frac{(x-x_{i-1})^r}{r!}y_v^{(r)}(x_{i-1})+O(h^{10})+(-1)^{v-1}\sum_{r=0}^{8}\frac{(x-x_{i-1})^r}{r!}y_{v-1}^{(r+1)}(x_{i-1})+O(h^9) \end{aligned}\quad, v=1,2,...$$

Thus, the local error is given by

$$\begin{aligned} &y_1(x)-g_1(x)\equiv O(h^{10}),\\ &y_2(x)-g_2(x)+y_1'(x)\equiv O(h^9),\\ &\quad\vdots\\ &y_v(x)-g_v(x)-(-1)^{v-1}y_{v-1}'(x)\equiv O(h^9) \end{aligned}$$

We observe that the NSCM applied to index-1 system is consistent of order **nine**, while it is consistent of order **eight** for index greater than one for all z_1, z_2, z_3, z_4 given in Table1.

Corollary 1 Let $y\in C^9[a,b]$ be Lipschitz continuous. Then the spline approximation $S(x)$ converges to the solution $y(x)$ as $h\to 0$ whenever (15) is fulfilled and

$$\lim_{h\to 0}S_0^{(j)}=y^{(j)}(x_0),\;\; j=0(1)4$$

Furthermore, the convergence order is nine for index-1 system, i.e., we have

$$\left| y\ (x_i) - S_i \right| \le C\,h^9, \quad i = 1(1)\mathrm{N} \tag{25a}$$

and convergence order is eight for index greater than or equal two, i.e., we have

$$\left| y\ (x_i) - S_i \right| \le C\,h^8, \quad i = 1(1)\mathrm{N} \tag{25b}$$

whenever the initial-values (7b) satisfy (25). In addition, the following global error estimate holds true:

$$\left| y(x) - S_i\ (x) \right| \le Ch^8, \quad x \in [a,b].$$

3. STRICT STABILITY

Before we can get started, we need the following definition.

Definition 2. The spline collocation method is strictly stable for the DAE (1.2) if the difference between perturbed spline collocation methods step,

$$F(t_{i-1+c_j}, W(t_{i-1+z_j}) + \delta_{v,i}^{(k)}, W'(t_{i-1+z_j})) = 0\ ,\ j,k = 1(1)5\ ,\ i = 1(1)\mathrm{N}\,, \tag{3.1}$$

where $W_0 = S_0 + \delta_0^{(0)}$, and $\left\| \delta_{v,i}^{(k)} \right\| \le \Delta_v$, $k = 0(1)5$, and unperturbed spline collocation methods step (1.10), satisfy $\left\| W(t_{i-1+c_j}) - S(t_{i-1+z_j}) \right\| \le K_0 \Delta_v$, j=1(1)4, i=1(1)N, where

$0 < h \le h_0$ and K_0, h_0 are constants depending only on the method and the DAEs.

We now solve (2.5) by the perturbed spline collocation method:

$$M\,W'(t_{i-1+z_j}) + W(t_{i-1+z_j}) - \delta_{v,i}^{(k)} = g(t_{i-1+z_j})\ ,\ j,k = 1(1)5,$$

where $W' = (w_1', w_2', \ldots, w_v')^T$, $W = (w_1, w_2, \ldots, w_v)^T$.

Then, we have

$$\underline{W}_{v,i} = \tilde{\mathrm{A}}_v \underline{W}_{v,i-1} + \mathrm{A}_v^{-1} \mathrm{D}_v\, \underline{g}_{v,i} + \underline{\delta}_{v,i}\,, \tag{3.2}$$

where the perturbations $\underline{\delta}_{v,i} = (\delta_{1,i}^{(1)}, \delta_{1,i}^{(2)}, \ldots, \delta_{v,i}^{(1)}, \delta_{v,i}^{(2)})^T$ satisfy $\left\| \underline{\delta}_{v,i} \right\| \le \Delta_v$,

$$\underline{W}_{v,i} = (W_{1,i}^{(1)}, W_{1,i}^{(2)}, \ldots, W_{v,i}^{(1)}, W_{v,i}^{(2)})^T,$$
$$\underline{W}_{v,i-1} = (W_{1,i-1}^{(1)}, W_{1,i-1}^{(2)}, \ldots, W_{v,i-1}^{(1)}, W_{v,i-1}^{(2)})^T.$$

Subtracting (3.2) from the corresponding expressions for the unperturbed solution (2.22), and letting $\underline{E}_{v,i} = \underline{W}_{v,i} - \underline{S}_{v,i}$, we obtain,

$$\underline{E}_{v,i} = \tilde{\mathrm{A}}_v \underline{E}_{v,i-1} + \underline{\delta}_{v,i}\ . \tag{3.3}$$

Using $\|.\|_\infty$, we have from (3.3)

$$\| \underline{E}_{v,i} \| \le R_v \| \underline{E}_{v,i-1} \| + \Delta_v\,, \tag{3.4}$$

where $R_\nu = \| \tilde{A}_\nu \|$ and $\| \underline{\delta}_{\nu,i} \| \le \Delta_\nu$,

Inequality (3.4) is defined recursively by

$$\| \underline{E}_{\nu,i} \| \le R_\nu^i \| \underline{E}_{\nu,0} \| + \sum_{k=0}^{i-1} R_\nu^{\ k} \Delta_\nu,\ i=1(1)N.$$

which can be rewritten in the form

$$\| \underline{E}_{\nu,i} \| \le R_\nu^{\ i} \| \underline{E}_{\nu,0} \| + \frac{1-R_\nu^{\ i}}{1-R_\nu} \Delta_\nu,\ i=1(1)N.$$

Note that $\lim_{N\to\infty} \frac{1-R_\nu^{\ N}}{1-R_\nu} = \frac{1}{1-R_\nu}$ if $R_\nu < 1$. Thus, we have the following theorem.

Theorem 3: The NSCM is strictly stable for index-v systems of DAEs iff:

$$R_\nu = \| \tilde{A}_\nu \|_\infty < 1. \qquad (3.5)$$

Proof. To prove that inequality (3.5) holds, we easily find that $R_\nu = \| \tilde{A}_\nu \|_\infty \equiv \| \tilde{A}_1 \|_\infty = \max_{1\le i\le 4} \sum_{j=1}^{4} | \tilde{a}_{i,j}^{1} |$, $v \ge 1$, where $\tilde{A}_1 = (\tilde{a}_{ij}^{1})$. Using Mathematia, we get the values of z_1, z_2, z_3, z_4 which satisfy the relation $R_\nu < 1$ in Table3. Moreover, for $R_\nu < 1$, we have $\lim_{i\to\infty} \| \underline{E}_{\nu,i} \| \le \| \underline{E}_{\nu,0} \| \lim_{i\to\infty} R_\nu^{\ i} + \Delta_\nu \lim_{i\to\infty} \frac{1-R_\nu^{\ i}}{1-R_\nu} = K_0 \Delta_\nu$,

where $K_0 = \frac{1}{1-R_\nu}$. This implies according to Definition 2 that the QSCM applied for index-v systems is strictly stable. □

Table 3: strictly stable intervals determined by some values of z_1, z_2, z_3, z_4

$z_1 = 0.6,\ z_2 = 0.99,\ z_3 = 0.999,\ z_4 = 0.9999$	$R_v = 0.671843$
$z_1 = 0.67,\ z_2 = 0.94,\ z_3 = 0.9999,\ z_4 = 0.99999$	$R_v = 0.735844$
$z_1 = 0.75,\ z_2 = 0.8,\ z_3 = 0.9999,\ z_4 = 0.99999$	$R_v = 0.973009$
$z_1 = 0.7, z_2 = 0.9, z_3 = 0.9999,\ z_4 = 0.99999$	$R_v = 0.684216$
$z_1 = 0.8, z_2 = 0.9, z_3 = 0.95, z_4 = 0.99$	$R_v = 0.426894$
$z_1 = 0.9,\ z_2 = 0.98,\ z_3 = 0.999,\ z_4 = 0.9999$	$R_v = 0.012811$

4. NUMERICAL RESULTS

In this section, four problems will be tested by using the spline method discussed above to demonstrate its efficiency for both linear and nonlinear problems. All computations where made with Turbo Pascal in double precision.

Problem 1: Consider the problem having four linear differential equations and one linear algebraic equation [7]

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$$\begin{aligned} y_1' &= -e^{x} y_1 + y_2 + y_4 + z - e^{-x} \quad , \\ y_2' &= -y_1 + y_2 - \sin(x) y_3 + z - \cos(x) \quad , \\ y_3' &= \sin(x) y_1 + y_3 + \sin(x) y_4 - \sin^2(x) - e^{-x}\sin(x) \ , \qquad x \in [0,\ 10] \\ y_4' &= \cos(x) y_2 + y_3 + \sin(x) y_4 - e^{-x}(1+\sin(x)) - \cos^2(x) - e^{x} \ , \\ 0 &= y_1 \sin^2(x) + y_2 \cos^2(x) + (y_3 - e^{x})(\sin(x) + 2\cos(x)) \\ &\quad + \sin(x)(y_4 - e^{-x})(\sin(x)+\cos(x)-1) - \sin^3(x) - \cos^3(x) \ . \end{aligned}$$

The exact solution to this system is $y_1 = \sin(x)$, $y_2 = \cos(x)$, $y_3 = e^{x}$, $y_4 = e^{-x}$, and $z(x) = e^{x}\sin(x)$. It is easy to verify that system is index-2 for all x. The results of our method with $z_1 = 0.8, z_2 = 0.9, z_3 = 0.95, z_4 = 0.99$ in Table 4 are compared with the results of quintic C^2- spline collocation method [7] in Table 5.

Table 4: The absolute error for Problem 1

x	The Presented Method, h=0.1.				
	δy_1	δy_2	δy_3	δy_4	δz
1.0	2.1E-16	6.1E-15	1.2E-15	1.4E-15	4.8E-15
2.0	9.9E-16	4.1E-15	9.4E-16	3.1E-15	7.2E-15
3.0	4.2E-16	1.3E-14	7.6E-15	5.4E-15	7.5E-16
4.0	3.3E-14	5.2E-13	5.2E-14	7.3E-14	1.7E-12
5.0	7.6E-14	8.8E-13	2.6E-13	3.4E-13	6.4E-12
6.0	4.5E-15	1.2E-12	6.6E-13	1.3E-14	1.0E-13
7.0	9.0E-15	6.3E-12	1.9E-12	2.0E-12	3.9E-12
8.0	2.4E-13	4.4E-11	9.8E-13	9.0E-12	7.3E-11
9.0	3.4E-15	2.7E-11	1.2E-11	7.6E-12	1.2E-11
10.0	1.8E-14	1.1E-10	2.7E-11	1.6E-11	2.6E-10

Table 5: The absolute error for Problem 1

x	Quintic C^2- Spline Collocation Method [7] (z_1=0.5, z_2=0.99), h=0.1.				
	δy_1	δy_2	δy_3	δy_4	δz
1.0	1.6254E-12	1.7312E-12	1.4800E-14	1.9253E-12	3.7635E-11
2.0	1.9271E-12	1.0742E-11	1.6891E-12	8.6127E-13	1.0713E-10
3.0	2.5514E-11	3.9746E-11	2.1647E-11	5.2040E-12	1.8240E-10
4.0	1.9124E-11	1.4622E-09	1.2677E-10	2.0503E-10	4.5873E-09
5.0	5.2136E-09	2.5915E-08	4.7350E-10	1.8017E-09	7.6844E-07
6.0	4.1878E-11	1.7231E-09	9.5689E-09	1.6561E-10	2.0305E-08
7.0	4.3640E-11	9.6056E-09	2.9558E-09	3.5497E-09	9.0305E-08
8.0	2.4179E-09	1.0691E-07	3.7785E-09	1.6413E-08	7.6251E-06
9.0	5.1095E-12	5.6267E-08	2.7056E-08	1.3849E-08	1.8266E-06
10.0	2.3257E-11	2.2294E-07	5.8476E-08	2.0865E-08	7.6798E-06

Problem 2: Consider the following differential algebraic equations with index-3 [**11**]

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} + \begin{pmatrix} 0 & 1 & x \\ 0 & 2 & 0 \\ 0 & x & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2x \\ e^{x} \end{pmatrix}, \qquad x \in [0,\ \infty],$$

The exact solution is $y_1 = e^{x} - 1$, $y_2 = 2x - e^{x}$, $y_3 = (1+x)e^{x} - 2x^2$. Table **6** appears comparisons between the absolute errors by our method and by using Pade approximation method in [**11**].

Table 6: The absolute errors for Problem 2

x	Tabatabaei and Celik [**11**]			**The Presented Method , h=0.1.** $z_1=0.8, z_2=0.9, z_3=0.95, z_4=0.99$		
	δy_1	δy_2	δy_3	δy_1	δy_2	δy_3
0.1	**1.0E-11**	**1.0E-11**	**1.0E-10**	**1.2 E-14**	**3.2 E-16**	**2.0 E-18**
0.2	**1.0E-10**	**1.0E-10**	**2.0E-09**	**2.3 E-14**	**3.0 E-16**	**1.1 E-17**
0.3	**1.0E-10**	**4.0E-10**	**1.0E-10**	**3.3 E-14**	**5.2 E-16**	**1.0 E-16**
0.4	**3.0E-10**	**4.0E-10**	**1.0E-09**	**3.4 E-13**	**7.3 E-16**	**3.4 E-16**
0.5	**3.0E-10**	**4.0E-10**	**1.0E-09**	**3.0 E-13**	**7.7 E-16**	**7.2 E-16**
0.6	**1.0E-10**	**6.0E-10**	**1.0E-09**	**4.5 E-14**	**1.2 E-16**	**1.3 E-17**
0.7	**3.0E-10**	**6.0E-10**	**3.0E-09**	**3.6 E-13**	**9.4 E-16**	**7.3 E-16**
0.8	**1.0E-11**	**4.0E-10**	**9.0E-09**	**2.2 E-13**	**5.1 E-16**	**5.4 E-16**
0.9	**5.0E-10**	**1.6E-09**	**3.0E-08**	**1.1 E-13**	**7.9 E-16**	**7.6 E-16**
1.0	**1.2E-09**	**4.0E-09**	**8.80E-08**	**6.3 E-15**	**1.0 E-16**	**1.0 E-16**
3.0	**-----**	**-----**	**-----**	**5.3 E-13**	**1.2 E-14**	**4.3 E-15**
6.0	**-----**	**-----**	**-----**	**3.9 E-11**	**1.3 E-13**	**8.3 E-14**
9.0	**-----**	**-----**	**-----**	**6.1 E-10**	**5.6 E-12**	**7.8 E-12**
10	**-----**	**-----**	**-----**	**9.2 E-10**	**8.9 E-12**	**5.1 E-11**

Problem 3: Consider index-2 Hessenberg DAE system with nonlinear differential equations and a nonlinear algebraic equation, as follows [**12**]:

$$y_1'' = y_1(4y_3 - 1) + 2(1-3t)y_2$$

$$y_2'' = 2\sin(y_3) + y_2(4y_4 - 1)$$

$$y_1^2 + t^2 y_2^2 - t^2 = 0 \ ,$$

subject to the initial conditions $y_1(0)=0,\ y_1'(0)=0,\ y_2(0)=1,\ y_2'(0)=0$. The exact solution is $y_3(t)=t(1-t),\ y_1(t)=t\sin(y_3(t)),\ y_2(t)=\cos(y_3(t))$. For comparison, we list in Table **7** the absolute errors by our method and by using Dhamacharoen method in [12]. Fig.1-3 explain the approximate spline solutions and the exact solution of y_1 , y_2 , y_3 by the presented method , for h=1/12.

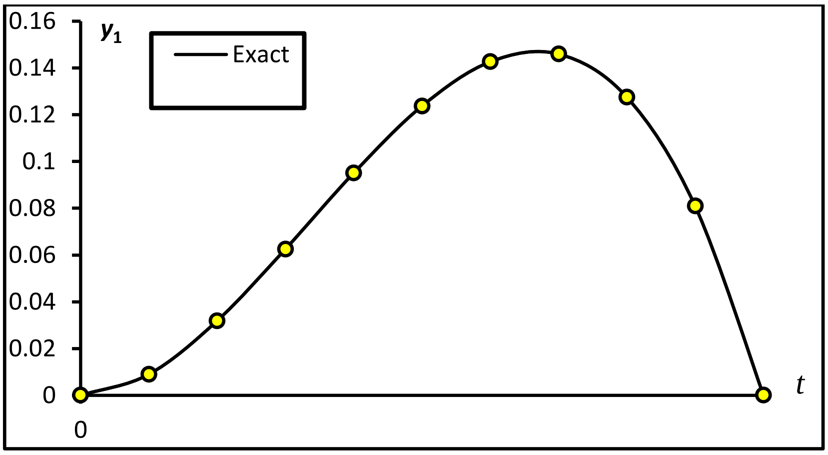

Fig.1: The spline solution S(x) and the exact solution y_1 , for Problem3, h=1/12.

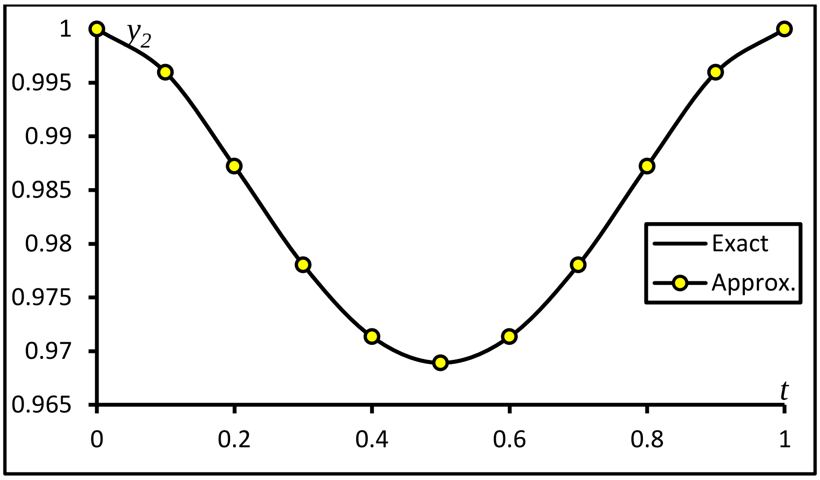

Fig.2: The spline solution S(x) and the exact solution y_2 for Problem3, h=1/12.

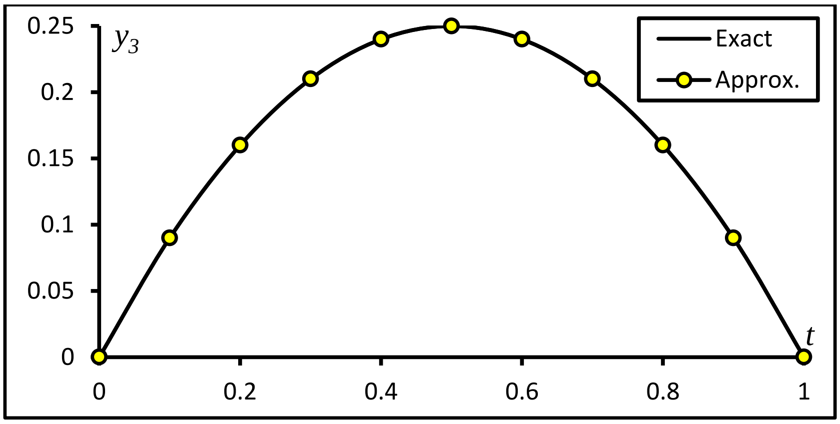

Fig.3. The spline solution and the exact solution y_3 for problem 3, for h=1/12.

Table 7: The absolute error for problem 3

t	Dhamacharoen method in [11]			Present spline method using the step size h=1/12		
	δy_1	δy_2	δy_3	δy_1	δy_2	δy_3
1/12	0.0000000	0.0000000	0.0028285	2.5 E-14	2.3 E-14	9.3 E-14
1/6	0.0000000	0.0000000	0.0027746	8.5 E-14	7.1 E-14	2.9 E-13
1/4	0.0000000	0.0000000	0.0028155	1.5 E-13	1.3 E-13	3.9 E-13
1/3	0.0000000	0.0000000	0.0027597	2.3 E-13	1.4 E-13	4.8 E-13
5/12	0.0000000	0.0000000	0.0027960	3.5 E-13	2.2 E-13	5.3 E-13
1/2	0.0000000	0.0000000	0.0027367	3.8 E-13	2.3 E-13	4.8 E-13
7/12	0.0000000	0.0000000	0.0027705	4.7 E-13	2.4 E-13	3.6 E-13
2/3	0.0000000	0.0000000	0.0027101	5.6 E-13	2.2 E-13	4.8 E-13
3/4	0.0000000	0.0000000	0.0027448	6.4 E-13	1.8 E-13	1.8 E-13
5/6	0.0000000	0.0000000	0.0026872	7.1 E-13	1.2 E-13	5.7 E-13
11/12	0.0000000	0.0000000	0.0027271	7.3 E-13	6.9 E-14	7.8 E-13
1	0.0000000	0.0000000	0.0026781	6.9 E-13	0.0 E+00	1.5 E-12

Problem 4: Consider the linear index-5 DAE

$$y_1' - y_2 = 0, \quad y_2' - y_3 = 0,$$

$$y_3' - y_4 = 0, \quad y_4' - y_5 = 0,$$

$$y_1 - Sin(x) = 0, \qquad t \in [0, 10],$$

subject to the initial conditions $y_1(0)=0$, $y_2(0)=1$, $y_3(0)=0$, $y_4(0)=-1$. The exact solution is $y_1(x)=sin(x)$, $y_2(x)=\cos(x)$, $y_3(x)=-\sin(x)$, $y_4(x)=-\cos(x)$, $y_5(x)=\mathrm{Sin}(x)$. We show the computational results in Table8. Fig.(1) explains the approximate spline solution and the exact solution of y_1 , y_2 , y_3, y_4 , y_5 by z_1=0.8, z_2=0.9, z_3:=0.966; z_4:=0.988 and h=0.4.

Table 8: The absolute error for problem 4 by present method

t	δy_1	δy_2	δy_3	δy_4	δy_5
0.40	0.0E+00	8.9E-17	8.2E-0014	2.1E-0011	2.8E-09
1.20	0.0E+00	2.9E-16	1.9E-0013	4.4E-0011	5.5E-09
2.00	0.0E+00	3.7E-16	2.0E-0013	4.5E-0011	7.0E-09
2.80	0.0E+00	2.3E-16	1.0E-0013	2.7E-0011	8.4E-09
3.60	0.0E+00	9.6E-17	5.1E-0014	4.4E-0012	1.2E-08
4.40	0.0E+00	2.8E-16	1.6E-0013	3.2E-0012	2.0E-08
5.20	0.0E+00	3.2E-16	1.4E-0013	1.8E-0011	3.5E-08
6.00	0.0E+00	1.2E-16	5.3E-0015	6.9E-0011	6.0E-08
6.80	0.0E+00	1.9E-16	1.9E-0013	1.3E-0010	7.6E-08
7.60	0.0E+00	3.5E-16	3.1E-0013	2.0E-0010	8.5E-08
8.40	0.0E+00	2.9E-16	3.2E-0013	2.6E-0010	9.0E-08
9.20	0.0E+00	2.9E-16	2.6E-0013	3.2E-0010	1.0E-07
10	0.0E+00	5.6E-17	1.8E-0013	4.1E-0010	2.1E-07

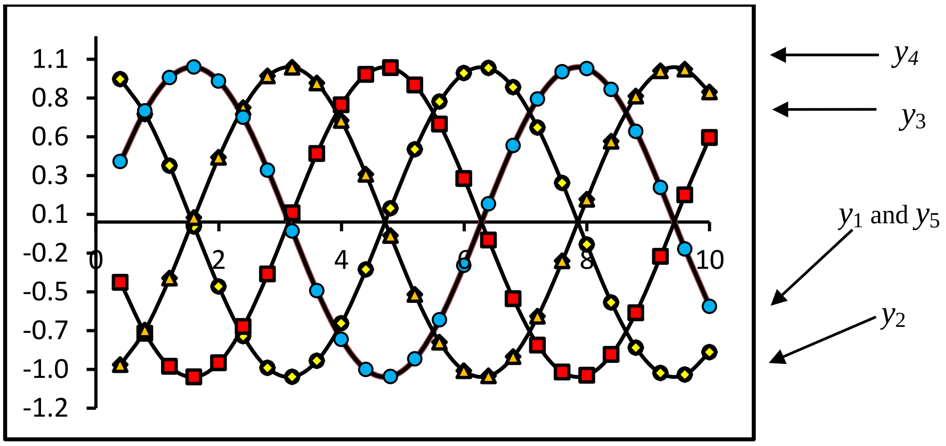

Fig.4. The approximate solution and the exact solution of y_1 , … , y_5 by z_1=0.8, z_2=0.9, z_3:=0.966; z_4:=0.988 and h=0.4, for problem 4.

5. CONCLUSIONS

The five point spline collocation method is successfully used for finding the approximate solutions of problems in higher index differential-algebraic equations**.** This method is tested on four linear and nonlinear problems and the results obtained are very encouraging and the purposed spline method performs better than the existing methods.

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