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A Study on Numerical Exact Solution of Euler, Improved Euler and Runge - Kutta Method

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Abstract: **In this paper, I will discuss the Runge Kutta method of solving simple linear differential equations. I start by starting why the Runge Kutta method is idea for solving simple linear differential equations numerically in comparison to move elementary methods. I will then proceed to explain what steps the method actually carries out in solving the differential equation along with the high level programming language code, I used to write a simple Runge Kutta solver and the output of the code, given some basic differential equations. I will then end by discussing the method is used to some examples to illustrate the accuracy and the implementation of the method.**

Keywords: **Euler, Improved Euler, Runge -Kutta.**

1. INTRODUCTION

[2]Carl David Runge(1856-1927), German mathematician and Physicist, worked for many years in spectroscopy. The analysis of data led him to consider problems in numerical computation, and the Runge Kutta method originated in his paper on the numerical solution of differential equations in 1895. The method was extended to system of equations in 1901 by M.Wilhelm Kutta (1867-1944). Kutta was a German mathematician and aerodynamicist who is also well known for his important contributions to classical airfoil theory.

In this work, We have develop High Level Programming Language to solve them to get exact value and approximated value by Euler, improved Euler and Runge Kutta fourth order method. We have then compared the accuracy of the results obtained by Heun's method(improved Euler method) and R-K fourth order method. We have also studied the whether the accuracy obtained by R-K fourth order method can be achieved by Heun's method by increasing the number of intervals.

[1] We consider the differential equation

$$y' = f(x,y) \qquad (1.1)$$

With the initial condition

$$y(x_0) = y_0$$

2. EULER METHOD:[1][2][3]

[3] The first attempt to solve a differential equation numerically was made by Euler about 1768.He used what is now called the tangent line method. Since x_0 & y_0 are known the slope of the line tangent to the solution at $t = t_0$,namely $\varphi'(t_0)=f(t_0, y_0)$, is also known. Hence we can construct the tangent line to the solution at t_0 and then obtain an approximate value y_1 of $\varphi(t_1)$ by moving along the tangent line from t_0 to t_1.

$$\text{Thus } y_1=y_0+\varphi'(t_0)(t_1-t_0)$$

$$=y_0+f(t_0,y_0)(t_1-t_0) \qquad (1.2)$$

Obtain an approximation value y2 of $\varphi(t_2)$ by moving along the tangent line from t1 to t_2.

$$y_2=y_1+f(t_1,y_1)(t_2-t_1) \qquad (1.3)$$

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Continuing in this manner,In general formula for the euler approximation is

$$y_{n+1}=y_n+f(t_n,y_n)(t_{n+1}-t_n) \quad (1.4)$$

We assume, $t_{n+1}=t_n+h$

We get $y_{n+1}=y_n+hf(t_n,y_n)$

$$=y_n+hf_n \quad (1.5)$$

Usually, the equ(1.5) is the better one by far, We discusses several algorithms that are vastly superior to the Euler method. In the meantime, It is easy to write a computer program to carry one the calculations required to produce the results in Table 1.

The outline of the programmed for euler method is given below.

STEP 1:**define** f(t,y)

STEP 2 :**input** initial value t_0 & y_0

STEP 3: **input** step size h and number of steps n

STEP 4: **output** t_0&y_0

STEP 5:for j from 1 to n do

STEP 6: k1 = f(t,y)

y = y+h*k_1

t = t+h

STEP 7: **output** t & y

STEP 8: **end**

Therefore, The calculation result can be displayed in graphical form, there are several excellent software packages that do this automatically, they can be very helpful in visualizing the behavior of solutions of differential equations.

Therefore, The analytical solution is not available if a numerical procedure is to be employed.

EXAMPLE 1:

Result for the numerical solution of y'=1-t+4y, y(0)=1.Using the Euler method for differential step sizes h.

Solution:

We consider the Euler formula as,

$$y_{n+1}=y_n+hf_n, n=0,1,2\ldots\ldots\ldots$$

To use high level programming language and derive the exact solution table in Euler method.

Table 1 Euler method

t	h=0.1	h=0.05	h=0.025	h=0.01	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5000000	1.5475000	1.5761188	1.5952901	1.6090418
0.2	2.1900000	2.3249000	2.4080117	2.4644587	2.5053299
0.3	3.1460000	3.4333560	3.6143837	3.7390345	3.8301388
0.4	4.4744000	5.0185326	5.3690304	5.6137120	5.7942260
0.5	6.3241600	7.2901870	7.9264062	8.3766865	8.7120041
0.6	8.9038240	10.550369	11.659058	12.454558	13.052522
0.7	12.505354	15.234032	17.112430	18.478797	19.515518
0.8	17.537495	21.967506	25.085110	27.384136	29.144880
0.9	24.572493	31.652708	36.746308	40.554208	43.497903
1.0	34.411490	45.588400	53.807866	60.037126	64.897803

3. IMPROVED EULER METHOD:

We have that the Euler method is not sufficiently accurate to be an efficient problem solving procedure, so this method is to develop more accurate methods.

Improved Euler method formula:

$$y_{n+1}=y_n+f_n+\frac{f(tn+h,yn+hfn)}{2}h \qquad (2.1)$$

By using computer programming code is to be the outline of the programmed is given below. In the meantime, It is easy to write a computer program to carry one the calculations required to produce the results in Table 2.

The Improved Euler method:

STEP 1:**define** f(t,y)

STEP 2 :**input** initial value t_0 & y_0

STEP 3: **input** step size h and number of steps n

STEP 4: **output** t_0&y_0

STEP 5:**for** j from 1 to n do

STEP 6: k_1 = f(t,y)

k_2 = f(t+h,y+h*k_1)

y=y+(h\2)*(k_1+k_2)

t = t+h

STEP 7: **output** t & y

STEP 8: **end**

EXAMPLE 2: A comparison of result using the linear and improved euler methods for the initial value problem y'=1-t-4y, y(0)=1.

Solution:

Table 2 Improved Euler Method

	Euler		Improved Euler		
t	h=0.05	h=0.025	h=0.1	h=0.05	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5475000	1.5761188	1.5950000	1.6049750	1.6090418
0.2	2.3249000	2.4080117	2.4636000	2.4932098	2.5053299
0.3	3.4333560	3.6143837	3.7371280	3.8030484	3.8301388
0.4	5.0185326	5.3690304	5.6099494	5.7404023	5.7942260
0.5	7.2901870	7.9264062	8.3697252	8.6117498	8.7120041
0.6	10.550369	11.659058	12.442193	12.873253	13.052522
0.7	15.234032	17.112430	18.457446	19.203865	19.515518
0.8	21.967506	25.085110	27.348020	28.614138	29.144880
0.9	31.652708	36.746308	40.494070	42.608178	43.497903
1.0	45.588400	53.807866	59.938223	63.424698	64.897803

This method gives the exact solution, the percentage errors at t=0.5 and t=1.0 are 1.15% & 2.3% respectively.

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4. THE RUNGE- KUTTA METHOD

We have introduced the Euler formula, the improved Euler formula, and the three term Taylor formula as ways to solve the initial value numerically as,

$y'=f(t,y)\ ,y(t_0)=y_0$ (3.1)

Therefore, the Runge- Kutta formula involves the value of f(t,y) at different points in the interval $t_n \le t \le t_{n+1.}$

It gives, $yn+1=yn+\frac{h}{6}(kn1+2kn2+2kn3+kn4)$ (3.2)

Where $k_{n1}=f(t_n,y_n)$, $kn2=f(t_n+\frac{1}{2}h, y_n+\frac{1}{2}hk_{n1})$, $k_{n3}=f(t_n+\frac{1}{2}h, y_n+\frac{1}{2}hk_{n2})$, $kn4=f(t_n+h,y_n+hk_{n3})$. (3.3)

The sum $(k_{n1}+2k_{n2}+2k_{n3}+k_{n4})/6$ is an average slope.

k_{n1}=slope at the left end of the interval

k_{n2}=slope at the midpoint using the Euler formula

k_{n3}=2nd approximation to the slope at the midpoint

k_{n4}=the slope at t_n+h using Euler formula and the slope k_{n3} to go from t_n to t_n+h.

Clearly the R-K formula equ (3.2) & (3.3), is more complicated than any of the formulas discussed previously .However, it is not hard to write computer program to implement this method. By using same structure as the algorithm for the Euler method.

Step 6 in Euler algorithm must be replaced as follows:

STEP 1:**define** f(t,y)

STEP 2 :**input** initial value t_0 & y_0

STEP 3: **input** step size h and number of steps n

STEP 4: **output** t_0&y_0

STEP 5:for j from 1 to n do

STEP 6: $k_1 = f(t,y)$

$k_2 =f(t+0.5*h,y+0.5*h*k_1)$

$k_3 = f(t+0.5*h,y+0.5*h*k_2)$

$k_4 =f(t+h,y+h*k_3)$

$y=y+(h\backslash 6)*(k_1+2*k_2+2*k_3+k_4)$

t = t+h

STEP 7: **output** t & y

STEP 8: **end**

EXAMPLE 3: A Comparison of result for the numerical solution of the initial value problem y'=1-t+4y, y(0)=1.

Solution: Taking h=0.2

we've $k_{01}=f(0,1)=5$; hk01=1.0

$k_{02}=f(0+0.1,1+0.5)=6.9$; hk02=1.38

$k_{03}=f(0+0.1,1+0.69)=7.66$; hk03=1.532

$k_{04}=f(0+0.2,1+1.532)=10.928$

Thus $y_1=1+\frac{0.2}{6}[5+2(6.9)+2(7.66)+10.928)]=1+1.5016=2.5016$

Table 3 Runge Kutta Method

	Euler	Improved Euler	Runge Kutta		
t	h=0.1	h=0.1	h=0.2	h=0.1	Exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5000000	1.5950000		1.6089333	1.6090418
0.2	2.1900000	2.4636000	2.5016000	2.5050062	2.5053299
0.3	3.1460000	3.7371280		3.8294145	3.8301388
0.4	4.4744000	5.6099494	5.7776358	5.7927853	5.7942260
0.5	6.3241600	8.3697252		8.7093175	8.7120041
0.6	8.9038240	12.442193	12.997178	13.047713	13.052522
0.7	12.505354	18.457446		19.507148	19.515518
0.8	17.537495	27.348020	28.980768	29.130609	29.144880
0.9	24.572493	40.494070		43.473954	43.497903
1.0	34.411490	59.938223	64.441579	64.858107	64.897803

Further result using the R-k method with h=0.2 and with h=0.1 give in Table 3.

Comparison of the value by Euler & Improved Euler methods are also in Table3.The R-K method yield a value at t=1.

(i) This t value differ from the exact solution by only 10%, the step size is h=0.2

(ii) 0.06% is the exact solution of h=0.1 only

Therefore, the R-K method with 10 steps is better than the improved euler method with 100 steps and also have that for h=0.01 the R-K method yields essentially the exact value of φ(1).

5. CONCLUSION

In this paper, We could successfully use the high level programming techniques which make the exact solution of the Runge Kutta method is easy fos us.It does not need thd difficult calculation.

REFERENCES

[1] B. Rai, D.P. Choudhury, H.I. Freedman, A Course in Ordinary Differential Equations,2002 Narosa Publishing House, Reprint 2004- Revised.

[2] William E.Boyce,Richard C.Diprima,Elementary Differential Equations and Boundary value problems,6th ed, 1997,by John Wiley & Sons,Inc.

[3] http://en.wikipedia.org/wiki/Runge-Kutta.