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A method to solve non-standard Bessel's equations

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Abstract: **A simple method to solve a specific type of Bessel's equations is proposed in this work. Several examples are given to illustrate the use of the proposed method and to demonstrate its simplicity. It has the advantage over the traditional approach for coming to an expedient solution in all circumstances.**

Keywords: **Bessel's equations, Several examples, expedient solution, circumstances.**

I. INTRODUCTION

Some differential equations encountered in engineering, despite of their similarity in appearance to Bessel's equations, are not of the standard forms. Such differential equations can be solved after they are transformed into standard Bessel's equations via proper variable substitutions. It is known that the general solutions of the *standard Bessel's equation of order n*

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \qquad (1)$$

can be expressed as $y(x) = c_1 J_n(x) + c_2 Y_n(x)$, where $J_n(x)$, $Y_n(x)$ are the Bessel function and the Neumann function of order *n*. The above result is generalized by Bowman[1] to

$$x^2 y'' + (1-2\alpha)xy' + \left(\beta^2\gamma^2 x^{2\gamma} + \alpha^2 - n^2\gamma^2\right)y = 0 \qquad (2)$$

whose general solution is known to be

$$y(x) = c_1 x^\alpha J_n\left(\beta x^\gamma\right) + c_2 x^\alpha Y_n\left(\beta x^\gamma\right) \qquad (3)$$

Here, we refer (2) is *non-standard Bessel's equation of first type*.

A special case of (2) , when $\alpha = 0$, is given by

$$x^2 y'' + xy' + \left[\left(\beta\gamma x^\gamma\right)^2 - n^2\gamma^2\right]y = 0 \qquad (4)$$

whose general solution is reduced to be

$$y(x) = c_1 J_n\left(\beta x^\gamma\right) + c_2 Y_n\left(\beta x^\gamma\right) \qquad (5)$$

Here, we refer (4) as *non-standard Bessel's equation of second type*.

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In many textbooks (e.g., by Peter V. O'Neil[2] or by Erwin Kreyszig[3]), a formula is given to solve this kind of equation. For example, you have to compare your equation with (2) or (4) to obtain the coefficients of α, β, γ and n, and only then the solution can be written by applying (3) or (5). In this work, we present a simple approach utilizing suitable substitution of variables directly so that the solution for standard Bessel's equation can be applied to obtain the solution. This method is still valid for Modified Bessel's equation.

II. NOTATIONS

$J_n(x)$: Bessel function of order n

$Y_n(x)$: Neumann function

III. RESULTS

Proposition 1.1. *The standard Bessel's equation of order n is equivalent to the non-standard Bessel equation of second type if and only if* $z = \beta x^\gamma$.

Proof. Let $z = \beta x^\gamma$ throughout the discussion. By chain rule of differentiation,

$$y' = \frac{dz}{dx}\frac{dy}{dz} = \beta\gamma x^{\gamma-1}\frac{dy}{dz}, \quad y'' = \beta\gamma(\gamma-1)x^{\gamma-2}\frac{dy}{dz} + \beta^2\gamma^2 x^{2\gamma-2}\frac{d^2y}{dz^2}$$

The above substitution shows that (4) holds if and only if

$$\beta^2\gamma^2 x^{2\gamma} y'' + \beta\gamma^2 x^\gamma y' + \left[\left(\beta\gamma x^\gamma\right)^2 - n^2\gamma^2\right] y = 0$$

which is equivalent to (1) after dividing out γ^2 . □

The following conclusion follows readily from the above proposition, and yet it proves to be a useful tip in finding solutions of Bessel's equations of various forms.

Corollary 1.1. *Bessel's equation of the form* $x^2 y'' + xy' + \left[\left(\sigma x^\gamma\right)^2 - \nu^2\right] y = 0$ *is equivalent to the standard Bessel's equation if and only if*

$$\frac{\sigma x^\gamma}{z} = \frac{\nu}{n} = \gamma \qquad (6)$$

where $\nu=0$ *mean* $n=0$.

Some examples are discussed here to illustrate our approach. First, consider the *non-standard Bessel's equation of second type*

$$x^2 y'' + xy' + \left(\frac{x}{4} - \frac{1}{16}\right) y = 0$$

Observe that $\frac{x}{4} - \frac{1}{16} = \left(\frac{\sqrt{x}}{2}\right)^2 - \left(\frac{1}{4}\right)^2$, one has

$$x^2 y'' + xy' + \left[\left(\frac{\sqrt{x}}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right] y = 0$$

According to Corollary 1.1. of (6), the equation can be transformed into (1) if

$$\frac{\frac{\sqrt{x}}{2}}{z}=\frac{\frac{1}{4}}{n}=\frac{1}{2}$$

i.e. $z=\sqrt{x}$, $n=\frac{1}{2}$. Hence the general solution $y=c_1J_{1/2}\left(\sqrt{x}\right)+c_2Y_{1/2}\left(\sqrt{x}\right)$ is obtained directly. Next, when $\nu=0$ is a *non-standard Bessel's equation of second type* equation such as $x^2y''+xy'+9x^4y=0$, one sets $9x^4=\left(3x^2\right)^2-0^2$. That is,

$$x^2y''+xy'+\left[\left(3x^2\right)^2-(0)^2\right]y=0$$

According to Corollary 1.1. of (6), the equation can be transformed into (1) if and only if

$$\frac{3x^2}{z}=\frac{0}{n}=2$$

i.e. $z=\frac{3}{2}x^2$, $n=0$. Hence the general solution is $y=c_1J_0\left(\frac{3}{2}x^2\right)+c_2Y_0\left(\frac{3}{2}x^2\right)$.

Moreover, consider a *non-standard Bessel's equation of first type* where the coefficient of y' is not *x*, such as $x^2y''-3xy'+4\left(x^4-3\right)y=0$, one can firstly transform it into (4) using $y=ux^{\delta}$, so that $x^2u''+(2\delta-3)xu'+\left(4x^4-(\delta^2-4\delta-12)\right)u=0$. Setting $\delta=2$, the resultant form gives the *non-standard Bessel's equation of second type* $x^2u''+xu'+\left((2x^2)^2-4^2\right)u=0$, which is in line with corollary 1.1. Renaming *y* and *u*, one yields

$$\frac{2x^2}{z}=\frac{4}{n}=2$$

i.e. $z=x^2$, $n=2$. This shows that $u=c_1J_2\left(x^2\right)+c_2Y_2\left(x^2\right)$; that is, the general solution of this equation is $y=c_1x^2J_2\left(x^2\right)+c_2x^2Y_2\left(x^2\right)$.

Finally, we consider the famous Airy's equation[3] as follows:

$$y''-xy=0$$

, which is belongs to the Modified Bessel's equation. Notice first that the coefficient of y' is not *x*. Hence let $y=ux^{\delta}$, so that $x^2u''+2\delta xu'-\left[x^3-\delta(\delta-1)\right]u=0$. Set $\delta=\frac{1}{2}$, then $x^2u''+xu'-\left(x^3+\frac{1}{4}\right)u=0$. Next, we write

$$x^2u''+xu'+\left[\left(ix^{3/2}\right)^2-\left(\frac{1}{2}\right)^2\right]u=0$$

which is in line with (4). Renaming *y* and *u*, one yields

$$\frac{ix^{3/2}}{z} = \frac{\frac{1}{2}}{n} = \frac{3}{2}$$

i.e. $z = \frac{2}{3} i x^{3/2}$, $n = \frac{1}{3}$. This shows that $u = c_1 J_{1/3}\left(\frac{2}{3} i x^{3/2}\right) + c_2 Y_{1/3}\left(\frac{2}{3} i x^{3/2}\right)$, and hence the general solution of the Airy's equation is $y = c_1 \sqrt{x} J_{1/3}\left(\frac{2}{3} i x^{3/2}\right) + c_2 \sqrt{x} Y_{1/3}\left(\frac{2}{3} i x^{3/2}\right)$.

IV. CONCLUSION

Previously, the only way to solve the non-standard or Modified Bessel's equation is troublesome, it can be solved only when the formula (3) is known, i.e., it cannot be solved directly. Our method provides a straightforward approach, which can be seen to be convenient and nimble through our examples. We believe the proposed method is a simple and useful technique to solve a variety of Bessel's equations in engineering and physics.

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