ACTION OF THE DIHEDRAL GROUP 
$D_5$ ON X= {1,2,3,4,5} 

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Abstract: Let $G$ be a group acting transitively on a set $X$, where $|X|= n$. Higman (1964) showed that if $G$ is of rank 3 and degree $n = \frac{k^2 + 1}{2}$, where $k$ is the length of of a $G_x$-orbit, $x \in X$ then $n = 5, 10, and 50$. He further show that such groups of degrees 5, 10 and 50 exist namely 

i). The dihedral group $D_5$ of order 10 has such a representation of degree 5, 

ii). The alternating group $A_5$ and the symmetric group $S_5$ acting on the set of unordered pairs of distinct points provide examples of degree 10, 

iii). $U_3(5)$ has rank 3 representation of degree 50 of this kind, as does the group obtained from it by adjoining the field automorphism. Degree 3250 remains undecided. 

In this project we shed more light on the groups in parts (i) and (ii). We study the subdegrees, suborbital graphs and intersection matrices corresponding to the representations of these groups. Properties of the suborbital graphs and intersection matrices associated with these representations will also be investigated.

1. PROBLEM STATEMENT 

In this project we are investigating the properties of rank 3 groups of degree $K^2 +1$. We intend to shed more light on the following groups: 

i). The dihedral group $D_5$ of order 10 with a representation of degree 5, 

ii). The alternating group $A_5$ and the symmetric group $S_5$ on 5 points acting on the set of unordered pairs of distinct points provide examples of degree 10, 

We deal with the following problems: 

i). Finding the subdegrees of these three groups. 

ii). Finding the suborbits and constructing suborbital graphs associated with the action and discussing the properties of these graphs. 

iii). Finding the intersection numbers associated with each non-trivial suborbit. 

iv). Finding the intersection matrix associated with each non-trivial suborbit and discussing the properties of these matrices.

2. OBJECTIVES 

Our main aim is to study the ranks, suborbits, subdegrees, suborbital graphs, intersection matrices corresponding to the action of the symmetric group $S_5$ and the alternating group $A_5$ on the set of unordered pairs from the set $X = \{ 1,2,3,4,5 \}$. We also study the action of the dihedral group $D_5$ on the set $X = \{ 1,2,3,4,5 \}$.
We intend to construct the suborbital graphs associated with the action and discuss their properties. And also compute the intersection numbers to come up with the intersection matrices.

3. INTRODUCTION

We investigate some properties of the action of $D_5$ on $X = \{1, 2, 3, 4, 5\}$. It is presented in three sections:

Section 1 deals with the subdegrees of $D_5$ acting on $X$

Section 2 deals with the suborbits of $D_5$ and the corresponding suborbital graphs.

Finally in section 3, we find the intersection numbers and the intersection matrices associated with each non-trivial suborbit.

Subdegrees of $G=D_5$ on $X = \{1, 2, 3, 4, 5\}$

Let $G$ act on the set $X = \{1, 2, 3, 4, 5\}$.

Lemma 1

$G$ acts transitively on $X = \{1, 2, 3, 4, 5\}$

Proof

Using the orbit-stabilizer theorem (Theorem 1.1.3.10), we need to show that the length of the orbit of a point say $1$ is five same as the number of points in $X$. This implies that the action of $G$ on $X$ has only one orbit.

Taking $1$ in $X$, $\text{Stab}_G(1) = \{1, (34)(25)\}$

Hence $|\text{Stab}_G(1)| = 2$

Applying the orbit–stabilizer theorem, we get

$$\text{Orbit}_G(1) = \left| \frac{G : \text{Stab}_G(1)}{\text{Stab}_G(1)} \right|$$

$$= \frac{|G|}{|\text{Stab}_G(1)|} = \frac{10}{2} = 5$$

Thus the orbit of $1 \in X$ is the whole of $X$. Therefore $G$ acts transitively on $X$.

Lemma 2

The number of orbits of $G_1$ on $X$ is 3.

Proof

To prove this we apply the Cauchy–Frobenius lemma (Theorem 1.1.3.8). We show that the number of orbits of the stabilizer of $1$ in $D_5$ is 3.

$\text{Stab}_{G_1}(1) = \{1, (25)(34)\} = G_1$

Hence

$\text{Orb}_{G_1}(1) = \{1\}$

$\text{Orb}_{G_1}(2) = \{2, 5\}$

$\text{Orb}_{G_1}(3) = \{3, 4\}$
Thus the number of orbits of $G_1$ on $X$ is 3 , implying that the rank of $G$ on $X$ is 3.

The three orbits of $G_1$ acting on $X$ determined above are :

$\Delta_0 = Orb_{G_1} (1) = \{1\}$. The trivial orbit

$\Delta_1 = Orb_{G_1} (2) = \{2,5\}$. The orbit containing 2 and 5

$\Delta_2 = Orb_{G_1} (3) = \{3,4\}$. The orbit containing 3 and 4

Therefore, the subdegrees of $D_5$ on $X$ are 1, 2 and 2.

Suborbital graphs of $G = D_5$

From the previous section , the suborbits of $G$ are :

$\Delta_0 = Orb_{G_1} (1) = \{1\} .

\Delta_1 = Orb_{G_1} (2) = \{2,5\} .

\Delta_2 = Orb_{G_1} (3) = \{3,4\} .

Now


By lemma 1.1.5.4 , we find that the suborbitals corresponding to the suborbits $\Delta_0, \Delta_1$ and $\Delta_2$ are:

$O_0(1,1) = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

$O_1(1,2) = \{(1,2), (2,3), (3,4), (4,5), (5,1), (1,5), (3,2), (5,4), (2,1), (4,3)\}$

$O_2(1,3) = \{(1,3), (2,4), (3,5), (4,1), (5,2), (1,4), (3,1), (5,3), (2,5), (4,2)\}$

From these suborbitals, we find the suborbital graphs. The suborbital graph corresponding to $O_0$ is the null graph.

We now consider the suborbital graphs corresponding to the suborbitals $O_1$ and $O_2$ respectively.

Since the order of $D_5$ is 10 which is even, by Theorem 1.1.5.6 [Wielandt , 1964 ,section 16.5] $\Delta_1$ and $\Delta_2$ are self-paired . Hence their corresponding suborbital graphs are undirected.

We then construct the suborbital graphs $\Gamma_1$ and $\Gamma_2$ in figure 5.2.1 and Figure 5.2.2 respectively.

The suborbital graph corresponding to the suborbit $\Delta_1$ of $G$ on $X$
\( \Gamma_1 \) is regular of degree 2. It is also connected.

The suborbital graph \( \Gamma_2 \) corresponding to the \( \Delta_2 \) of \( G \) on \( X \).

\[ 1 \quad 2 \quad 3 \]
\[ 5 \quad 4 \]

\( \Gamma_2 \) is regular of degree 2 and is connected.

3. Intersection matrices associated with the action of \( G=D_5 \) on \( X \)

In this section we compute the intersection numbers and the corresponding intersection matrices associated with each non-trivial suborbit \( \Delta_1 \) and \( \Delta_2 \).

By Definition 1.1.6.1, given an arrangement of the \( G_e \)-orbits, the \( G_b \)-orbits are arranged such that if \( b \in X \) and \( g(a) = b \) then,

\[ g(\Delta_i(a)) = \Delta_i(g(b)) = \Delta_i(b) \]

**Intersection matrix corresponding to** \( \Delta_1(1) \)

From a general discussion of intersection numbers and intersection matrices in section 1.1.6.

Now taking \( a = 1 \) in \( X \) and \( G_1 \)-orbits arranged as follows,

\[ \Delta_0(1) = \{1\} \]
\[ \Delta_1(1) = \{2,5\} \]
\[ \Delta_2(1) = \{3,4\} \]

We arrange the \( G_5 \) orbits as follows:

\[ \Delta_0(2) = \{2\} \]
\[ \Delta_1(2) = \{1,3\} \]
\[ \Delta_2(2) = \{4,5\} \]
\( \Delta_0 (3) = \{3\} \).
\( \Delta_1 (3) = \{1,5\} \).
\( \Delta_2 (3) = \{2,4\} \).

From definition 1.1.6.1, the intersection numbers relative to the suborbit \( \Delta_1 (1) \) are defined by

\[
\mu^{(i)}_j = |\Delta_i (b) I \Delta_j (1)|, \quad b \in \Delta_j (1),
\]

Hence we find the intersection numbers relative to \( \Delta_1 (1) \) as follows

\[
\mu_0^{(i)} = |\Delta_1 (1) I \Delta_0 (1)| = 0 \\
\mu_1^{(i)} = |\Delta_1 (1) I \Delta_1 (1)| = 2 \\
\mu_2^{(i)} = |\Delta_1 (1) I \Delta_2 (1)| = 0 \\
\mu_0^{(i)} = |\Delta_1 (2) I \Delta_0 (1)| = 1 \\
\mu_1^{(i)} = |\Delta_1 (2) I \Delta_1 (1)| = 0 \\
\mu_2^{(i)} = |\Delta_1 (2) I \Delta_2 (1)| = 1 \\
\mu_0^{(i)} = |\Delta_1 (3) I \Delta_0 (1)| = 1 \\
\mu_1^{(i)} = |\Delta_1 (3) I \Delta_1 (1)| = 1 \\
\mu_2^{(i)} = |\Delta_1 (3) I \Delta_2 (1)| = 0
\]

By definition 1.1.6.2 the intersection matrix \( M_1 = (\mu^{(i)}_{j,k}) \), associated with \( \Delta_1 \{1,2\} \) where \( \mu^{(i)}_{j,k} \) are the intersection numbers relative to \( \Delta_1 (1) \) is obtained as follows:

\[
M_1 = \begin{bmatrix}
\mu_0^{(i)} & \mu_0^{(i)} & \mu_0^{(i)} \\
\mu_1^{(i)} & \mu_1^{(i)} & \mu_1^{(i)} \\
\mu_2^{(i)} & \mu_2^{(i)} & \mu_2^{(i)}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 1 \\
2 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

Intersection matrix corresponding to \( \Delta_2 (1) \)

From definition 1.1.6.1, the intersection numbers relative to the suborbit \( \Delta_2 (1) \) are defined by

\[
\mu^{(i)}_j = |\Delta_2 (b) I \Delta_j (1)|, \quad b \in \Delta_j (1),
\]

We therefore find the intersection numbers relative to \( \Delta_2 (1) \) as follows
\[
\mu_{00}^{(2)} = |\Delta_1(1) I \Delta_0(1)| = 0
\]
\[
\mu_{10}^{(2)} = |\Delta_2(1) I \Delta_1(1)| = 0
\]
\[
\mu_{20}^{(2)} = |\Delta_2(1) I \Delta_2(1)| = 2
\]
\[
\mu_{01}^{(2)} = |\Delta_2(2) I \Delta_0(1)| = 0
\]
\[
\mu_{11}^{(2)} = |\Delta_2(2) I \Delta_1(1)| = 1
\]
\[
\mu_{21}^{(2)} = |\Delta_2(2) I \Delta_2(1)| = 1
\]
\[
\mu_{02}^{(2)} = |\Delta_2(3) I \Delta_0(1)| = 0
\]
\[
\mu_{12}^{(2)} = |\Delta_2(3) I \Delta_1(1)| = 1
\]
\[
\mu_{22}^{(2)} = |\Delta_2(3) I \Delta_2(1)| = 1
\]

By definition 1.1.6.2 the intersection matrix \( M_2 = \left( \mu_{ij}^{(2)} \right) \), associated with \( \Delta_2(1) \) where \( \mu_{ij}^{(2)} \) are the intersection numbers relative to \( \Delta_2(1) \) is obtained as follows;

\[
M_2 = \begin{bmatrix}
\mu_{00}^{(2)} & \mu_{01}^{(2)} & \mu_{02}^{(2)} \\
\mu_{10}^{(2)} & \mu_{11}^{(2)} & \mu_{12}^{(2)} \\
\mu_{20}^{(2)} & \mu_{21}^{(2)} & \mu_{22}^{(2)} \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}
\]

**Properties of the intersection matrices associated with \( \Delta_1(1) \) and \( \Delta_2(1) \)**

By computation of the intersection matrices we are able to come up with the following properties.

i). The column sum of the intersection matrix associated with \( \Delta_i \) is equal to the degree (valency) of the suborbital graph corresponding to the same suborbit \( \Delta_i \), which is also the length of the suborbit.

We can see that the column sum of \( M_1 \) is 2 equal to the degree of \( \Gamma_1 \). Also the column sum of \( M_2 \) is 2 equal to the degree of \( \Gamma_2 \).

ii). \( M_1 \) and \( M_2 \) are square matrices.

iii). The order of \( M_1 \) and \( M_2 \) is \( 3 \times 3 \) since the rank of \( D_3 \) is 3.

4. **CONCLUSION**

In this project we investigated some properties of the action of \( D_5 \) on \( X = \{1, 2, 3, 4, 5\} \), we showed that \( D_3 \) acts transitively on \( X \). We found the rank of \( D_3 \) when it acts on \( X \) to be 3, same as that obtained by Higman (1964). And that the subdegrees of \( D_3 \) are 1, 2 and 2. We also constructed the suborbital graphs and found out that suborbital graphs \( \Gamma_1 \)
and \( \Gamma_2 \) corresponding to the non-trivial suborbits of \( D_4 \) are regular and undirected. We computed the intersection numbers and intersection matrices associated with each non-trivial suborbit \( \Delta_1 \) and \( \Delta_2 \). We found out that the intersection matrices \( M_1 \) and \( M_2 \) are square matrices and that the column sum of \( M_1 \) is 2 equal to the degree of \( \Gamma_1 \) and the column sum of \( M_2 \) is also 2 equal to the degree of \( \Gamma_2 \).

REFERENCES


