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Another Representation of Fractional Exponential Function and Fractional Logarithmic Function

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***Abstract*: This paper gives another representation of general fractional exponential function and fractional logarithmic function. In addition, we discuss some properties of them based on Jumarie type of Riemann-Liouville (R-L) fractional calculus. These properties are the same as those of classical exponential function and logarithmic function. The main methods used in this paper are the chain rule for fractional derivatives and a new multiplication of fractional analytic functions.**

***Keywords*: Representation, Fractional exponential function, Fractional logarithmic function, Jumarie type of R-L fractional calculus, Chain rule for fractional derivatives, New multiplication, Fractional analytic functions.**

I. INTRODUCTION

After the long-term unremitting efforts of many scholars, fractional calculus theory has been established to a certain extent. With the development of computer technology, fractional calculus is widely used in various fields of science and engineering, such as physics, mechanics, signal processing, viscoelasticity, economics, bioengineering, and control [1-7]. At present, the definitions of fractional calculus mainly include Riemann-Liouville (R-L) type, Caputo type, Grunwald-Letnikov (G-L) type, Weyl type, Riesz type, Jumarie type, etc [8-11].

In this paper, we provide another representation of general fractional exponential function and fractional logarithmic function. Moreover, several properties of them are obtained based on Jumarie's modified R-L fractional calculus. A new multiplication of fractional analytic functions, and the chain rule for fractional derivatives play important roles in this paper. In fact, the results obtained in this article are generalizations of those in traditional calculus.

II. PRELIMINARIES

The fractional calculus used in this article and some properties are introduced below.

Definition 2.1 ([12])**:** Suppose that $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie′s modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt. \tag{1}$$

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And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt, \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([13])**:** *Let* α, β, x_0, C *be real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Next, we introduce the fractional analytic function.

Definition 2.3 ([14])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, an α-fractional power series on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Moreover, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([15])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{8}$$

Definition 2.5 ([15])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

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$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\big(g_\alpha(x^\alpha)\big)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\big(f_\alpha(x^\alpha)\big)^{\otimes k}. \tag{12}$$

Definition 2.6 ([15])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions at x_0 satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

The followings are some fractional analytic functions.

Definition 2.7([16])**:** If $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. In addition, the α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

The main methods used in this paper are introduced below.

Theorem 2.8: (chain rule for fractional derivatives) ([16])**:** *Assume that* $0 < \alpha \le 1, x_0$ *is a real number, and* $f_\alpha(x^\alpha), g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions at* $x = x_0$. *Then*

$$\left({}_{x_0}D_x^\alpha\right)\left[f_\alpha\big(g_\alpha(x^\alpha)\big)\right] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)]\big(g_\alpha(x^\alpha)\big) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{17}$$

Definition 2.9 ([16])**:** Suppose that $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)}$ is defined by

$$f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)} = E_\alpha\left(g_\alpha(x^\alpha) \otimes Ln_\alpha\big(f_\alpha(x^\alpha)\big)\right). \tag{18}$$

Theorem 2.10 ([17])**:** *Assume that* $0 < \alpha \le 1$, *and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are two* α*-fractional analytic functions at* $x = x_0$. *Then the* α*-fractional derivative of the* α*-fractional power exponential function* $f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)}$ *is*

$$\left({}_{x_0}D_x^\alpha\right)\left[f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)}\right] = f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)} \otimes \left(\begin{array}{c}\left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)] \otimes Ln_\alpha\big(f_\alpha(x^\alpha)\big) \\ +g_\alpha(x^\alpha) \otimes f_\alpha(x^\alpha)^{\otimes -1} \otimes \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)]\end{array}\right). \tag{19}$$

III. RESULTS AND PROPERTIES

Definition 3.1: Let $0 < \alpha \le 1$, and $a_\alpha > 0, a_\alpha \ne 1$. Then

$$a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)}x^\alpha} = E_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes Ln_\alpha(a_\alpha)\right) = E_\alpha\left(Ln_\alpha(a_\alpha) \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \tag{20}$$

is called the α-fractional exponential function based on a_α .

Definition 3.2: Let $0 < \alpha \le 1$. We define

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$$e_\alpha = E_\alpha(1) = 1 + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)} + \cdots = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} . \tag{21}$$

Proposition 3.3: *Suppose that* $0 < \alpha \leq 1$. *Then*

$$Ln_\alpha(e_\alpha) = 1. \tag{22}$$

And

$$E_\alpha(x^\alpha) = e_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}. \tag{23}$$

Proof $Ln_\alpha(e_\alpha) = Ln_\alpha\big(E_\alpha(1)\big) = 1$. Thus

$$e_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha} = E_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes Ln_\alpha(e_\alpha)\right) = E_\alpha(x^\alpha) .$$ Q.e.d.

Theorem 3.4: *If* $0 < \alpha \leq 1$, *and* $a_\alpha > 0$, $a_\alpha \neq 1$. *Then the* α*-fractional derivative of* $a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}$,

$$\left({}_0D_x^\alpha\right)\left[a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}\right] = Ln_\alpha(a_\alpha) \cdot a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}. \tag{24}$$

Proof Using Theorem 2.10 yields the desired result holds. Q.e.d.

Corollary 3.5: *Assume that* $0 < \alpha \leq 1$, *then*

$$\left({}_0D_x^\alpha\right)\left[e_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}\right] = e_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}. \tag{25}$$

Definition 3.6: Let $0 < \alpha \leq 1$, and $a_\alpha > 0, a_\alpha \neq 1$. Then we define $Log_{a_\alpha}(x^\alpha)$ is the inverse function of $a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}$. In particular, $Log_{e_\alpha}(x^\alpha) = Ln_\alpha(x^\alpha)$.

Theorem 3.7: *Assume that* $0 < \alpha \leq 1$, *and* $a_\alpha > 0$, $a_\alpha \neq 1$. *Then the* α*-fractional derivative of* $Log_{a_\alpha}(x^\alpha)$,

$$\left({}_0D_x^\alpha\right)\left[Log_{a_\alpha}(x^\alpha)\right] = \frac{1}{Ln_\alpha(a_\alpha)} \cdot \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]^{\otimes -1}. \tag{26}$$

Proof Since $a_\alpha^{\otimes Log_{a_\alpha}(x^\alpha)} = \frac{1}{\Gamma(\alpha+1)} x^\alpha$, it follows that

$$\left({}_0D_x^\alpha\right)\left[a_\alpha^{\otimes Log_{a_\alpha}(x^\alpha)}\right] = 1. \tag{27}$$

On the other hand, by Theorem 2.10, we have

$$\begin{aligned}\left({}_0D_x^\alpha\right)\left[a_\alpha^{\otimes Log_{a_\alpha}(x^\alpha)}\right] &= Ln_\alpha(a_\alpha) \cdot a_\alpha^{\otimes Log_{a_\alpha}(x^\alpha)} \otimes \left({}_0D_x^\alpha\right)\left[Log_{a_\alpha}(x^\alpha)\right] \\ &= Ln_\alpha(a_\alpha) \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)\left[Log_{a_\alpha}(x^\alpha)\right].\end{aligned} \tag{28}$$

Therefore,

$$\left({}_0D_x^\alpha\right)\left[Log_{a_\alpha}(x^\alpha)\right] = \frac{1}{Ln_\alpha(a_\alpha)} \cdot \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]^{\otimes -1} .$$ Q.e.d.

Proposition 3.8: *Let* $0 < \alpha \leq 1$, *and* $a_\alpha > 0$, $a_\alpha \neq 1$. *Then*

$$a_\alpha^{\otimes\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha + \frac{1}{\Gamma(\alpha+1)} y^\alpha\right)} = a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha} \otimes a_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} y^\alpha}. \tag{29}$$

$$Log_{a_\alpha}(x^\alpha \otimes y^\alpha) = Log_{a_\alpha}(x^\alpha) + Log_{a_\alpha}(y^\alpha). \tag{30}$$

Proof By Definition 3.1,

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$$a_\alpha^{\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)}=E_\alpha\left(Ln_\alpha(a_\alpha)\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)\right)$$
$$=E_\alpha\left(Ln_\alpha(a_\alpha)\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes E_\alpha\left(Ln_\alpha(a_\alpha)\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$
$$=a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha}\otimes a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}y^\alpha}.$$

On the other hand, since

$$a_\alpha^{\otimes\left(Log_{a_\alpha}(x^\alpha\otimes y^\alpha)\right)}=\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\frac{1}{\Gamma(\alpha+1)}y^\alpha. \tag{31}$$

And

$$a_\alpha^{\otimes\left(Log_{a_\alpha}(x^\alpha)+Log_{a_\alpha}(y^\alpha)\right)}$$
$$=a_\alpha^{\otimes Log_{a_\alpha}(x^\alpha)}\otimes a_\alpha^{\otimes Log_{a_\alpha}(y^\alpha)}$$
$$=\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes\frac{1}{\Gamma(\alpha+1)}y^\alpha\,. \tag{32}$$

It follows that

$$Log_{a_\alpha}(x^\alpha\otimes y^\alpha)=Log_{a_\alpha}(x^\alpha)+Log_{a_\alpha}(y^\alpha).$$ Q.e.d.

IV. CONCLUSION

In this article, we obtain another representation of general fractional exponential function and fractional logarithmic function. In addition, based on Jumarie modification of R-L fractional calculus, some properties of them are discussed. A new multiplication and the chain rule for fractional derivatives play important roles in this paper. In fact, the results we obtained are generalizations of those of classical exponential function and logarithmic function. On the other hand, the new multiplication is a natural operation of fractional analytic functions. In the future, we will use fractional exponential function and logarithmic function to study the problems in engineering mathematics and fractional differential equations.

REFERENCES

[1] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[2] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348, Springer, Wien, Germany, 1997.

[3] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, Mechanical Engineering and Solid Mechanics, Wiley-ISTE, Croydon, 2014.

[4] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[5] R. L. Magin, Fractional calculus in bioengineering, 13th International Carpathian Control Conference, 2012.

[6] N. Sebaa, Z. E. A. Fellah, W. Lauriks, C. Depollier, Application of fractional calculus to ultrasonic wave propagation in human cancellous bone, Signal Processing, vol. 86, no. 10, pp. 2668-2677, 2006.

[7] L. Debnath, Recent applications of fractional calculus to science and engineering. International Journal of Mathematics and Mathematical Sciences, vol. 54, pp. 3413-3442, 2003.

[8] S. Das, Functional fractional calculus. 2nd ed. Springer-Verlag, 2011.

[9] K. Diethelm, The analysis of fractional differential equations. Springer-Verlag, 2010.

[10] I. Podlubny, Fractional differential equations. San Diego: Academic Press; 1999.

[11] K. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations. New York: Wiley, 1993.

[12] C. -H. Yu, Using trigonometric substitution method to solve some fractional integral problems, International Journal of Recent Research in Mathematics Computer Science and Information Technology, vol. 9, no. 1, pp. 10-15, 2022.

[13] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[14] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[15] C. -H. Yu, A study on arc length of nondifferentiable curves, Research Inventy: International Journal of Engineering and Science, vol. 12, no. 4, pp. 18-23, 2022.

[16] C. -H. Yu, Research on fractional exponential function and logarithmic function, International Journal of Novel Research in Interdisciplinary Studies, vol. 9, no. 2, pp. 7-12, 2022.

[17] C. -H. Yu, A study on fractional derivative of fractional power exponential function, American Journal of Engineering Research, vol. 11, no. 5, pp. 100-103, 2022.