Arbitrary Order Fractional Derivative of Inverse Fractional Trigonometric Function

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Abstract: In this paper, based on Jumarie’s modified Riemann-Liouville (R-L) fractional derivative, any order fractional derivative of inverse fractional trigonometric function is obtained. The fractional binomial series and a new multiplication of fractional analytic functions play important roles in this article. In fact, these results we obtained are generalizations of those in traditional calculus. Moreover, the new multiplication is a natural operation of fractional analytic functions.

Keywords: Jumarie’s modified R-L fractional derivative, Inverse fractional trigonometric function, Fractional binomial series, New multiplication, Fractional analytic functions.

I. INTRODUCTION

In applied mathematics and mathematical analysis, fractional calculus theory is used to deal with any real or complex order of derivatives or integrals. Its first appearance is in a letter written to L’Hôpital by Leibniz in 1695. Over the years, many mathematicians have used their own symbols and methods to find various definitions that conform to the concept of non integer derivative or integral. The definitions of fractional calculus mainly include Riemann-Liouville (R-L) type, Caputo type, Grunwald-Letnikov (G-L) type, Weyl type, Riesz type, Jumarie type, etc [1-5]. Fractional calculus provides a good tool to describe physical memory and heredity. Fractional calculus has been applied to many fields such as biological materials, control and robotics, viscoelastic dynamics, chaos, and quantum mechanics. Those applications have also accelerated the development of the theory of fractional calculus [6-11].

Based on Jumarie type of R-L fractional derivative, this paper obtained arbitrary order fractional derivative of inverse fractional trigonometric function. The major methods we used are the fractional binomial series and a new multiplication of fractional analytic functions. And the results obtained in this article are generalizations of the results in ordinary calculus.

II. DEFINITIONS AND PROPERTIES

The fractional calculus used in this study and some properties are introduced below.

Definition 2.1 ([12]): Suppose that 0 < α ≤ 1, and x₀ is a real number. The Jumarie’s modified R-L α-fractional derivative is defined by

\[
(x₀Dₜ^α)[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx₀} \int_{x₀}^{x} \frac{f(t)-f(x₀)}{(x-t)^α} dt.
\]

(1)

And the Jumarie type of R-L α-fractional integral is defined by

\[
(x₀Iₜ^α)[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{x₀}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} dt,
\]

(2)

where \( \Gamma(\cdot) \) is the gamma function. In addition, we define \((x₀Dₜ^α)^n[f(x)] = (x₀Dₜ^α)(x₀Dₜ^α)^{n-1}[f(x)],[x₀Dₜ^α]…(x₀Dₜ^α)[f(x)]\), and it is called the n-th order \( \alpha \)-fractional derivative of \( f(x) \), where \( n \) is any positive integer.
Proposition 2.2 ([13]): Let \( \alpha, \beta, x_0, C \) be real numbers and \( \beta \geq \alpha > 0 \), then
\[
\left( x_0 D_\alpha^\beta \right)[(x - x_0)^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x - x_0)^{\beta-\alpha},
\] (3)
and
\[
\left( x_0 D_\alpha^\beta \right)[C] = 0.
\] (4)
The following is the definition of fractional power series expansion of fractional function.

Definition 2.3 ([14]): Let \( x, x_0 \) and \( c_k \) be real numbers for all \( k \), and \( 0 < \alpha \leq 1 \). The series \( \sum_{k=0}^\infty c_k (x - x_0)^{\alpha k} \) is called a real \( \alpha \)-fractional power series. Its disk of convergence intersects the real axis in an interval \( (x_0 - s, x_0 + s) \) called the interval of convergence. Each real \( \alpha \)-fractional power series defines a real valued sum function whose value at each \( x \) in the interval of convergence is given by
\[
f_\alpha(x^\alpha) = \sum_{k=0}^\infty c_k (x - x_0)^{\alpha k}.
\] (5)
This series is said to represent \( f_\alpha \) in the interval of convergence, and it is called a \( \alpha \)-fractional power series expansion of \( f_\alpha \) about \( x_0 \).

Definition 2.4 ([14]): Let \( 0 < \alpha \leq 1 \) and \( f_\alpha \) be a real valued \( \alpha \)-fractional function defined on an interval \( I \) contained in \( \mathbb{R} \). If \( f_\alpha \) has \( \alpha \)-fractional derivatives of every order at each point of \( I \), we write \( f_\alpha \in C_\alpha^\infty(I) \). If \( f_\alpha \in C_\alpha^\infty(I) \) on some neighborhood of a point \( x_0 \), the series
\[
\sum_{k=0}^\infty \frac{(x_0 - x)^{\alpha k}}{k\Gamma(\alpha k+1)} f(x)(x_0)^\alpha
\] (6)
is called the \( \alpha \)-fractional Taylor series about \( x_0 \) generated by \( f_\alpha \). To indicate that \( f_\alpha \) generate this fractional Taylor series, we write
\[
f_\alpha(x^\alpha) \sim \sum_{k=0}^\infty \frac{(x_0 - x)^{\alpha k}}{k\Gamma(\alpha k+1)} f(x)(x_0)^\alpha.
\] (7)

Theorem 2.5 ([14]): Suppose that \( 0 < \alpha \leq 1 \), and \( f_\alpha(x^\alpha) = \sum_{k=0}^\infty c_k (x - x_0)^{\alpha k} \). Then
\[
f_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{(x_0 - x)^{\alpha k}}{k\Gamma(\alpha k+1)} f(x)(x_0)^\alpha.
\] (8)
Next, a new multiplication of fractional analytic functions is introduced.

Definition 2.6 ([15]): If \( 0 < \alpha \leq 1 \), and \( x_0 \) is a real number. Let \( f_\alpha(x^\alpha) \) and \( g_\alpha(x^\alpha) \) be two \( \alpha \)-fractional analytic functions defined on an interval containing \( x_0 \),
\[
f_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{a_k}{k!} (x - x_0)^{\alpha k} = \sum_{k=0}^\infty \frac{a_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^\alpha
\] (9)
and
\[
g_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{b_k}{k!} (x - x_0)^{\alpha k} = \sum_{k=0}^\infty \frac{b_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^\alpha.
\] (10)
Then
\[
f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{a_k}{k!} \frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \otimes \sum_{k=0}^\infty \frac{b_k}{k!} (x - x_0)^\alpha
\] (11)
Equivalently,
\[
f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{a_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^\alpha \otimes \sum_{k=0}^\infty \frac{b_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^\alpha.
\] (12)
Definition 2.7 ([15]): Let $0 < \alpha \leq 1$, and $f_{\alpha}(x^\alpha)$, $g_{\alpha}(x^\alpha)$ be two $\alpha$-fractional analytic functions defined on an interval containing $x_0$.

\[
f_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(\alpha k+1)} (x-x_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (x-x_0)^\alpha \right)^{\alpha k},
\]

(13)

\[
g_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(\alpha k+1)} (x-x_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left( \frac{1}{\Gamma(\alpha+1)} (x-x_0)^\alpha \right)^{\alpha k}.
\]

(14)

The compositions of $f_{\alpha}(x^\alpha)$ and $g_{\alpha}(x^\alpha)$ are defined by

\[
(f_{\alpha} \circ g_{\alpha})(x^\alpha) = f_{\alpha}(g_{\alpha}(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left( g_{\alpha}(x^\alpha) \right)^{\alpha k},
\]

(15)

and

\[
(g_{\alpha} \circ f_{\alpha})(x^\alpha) = g_{\alpha}(f_{\alpha}(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left( f_{\alpha}(x^\alpha) \right)^{\alpha k}.
\]

(16)

Definition 2.8 ([15]): Let $0 < \alpha \leq 1$. If $f_{\alpha}(x^\alpha)$, $g_{\alpha}(x^\alpha)$ are two $\alpha$-fractional analytic functions at $x_0$ satisfies

\[
(f_{\alpha} \circ g_{\alpha})(x^\alpha) = (g_{\alpha} \circ f_{\alpha})(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} (x-x_0)^\alpha.
\]

Then $f_{\alpha}(x^\alpha)$, $g_{\alpha}(x^\alpha)$ are called inverse functions of each other.

The followings are some fractional analytic functions.

Definition 2.9 ([16]): If $0 < \alpha \leq 1$, $x$ is a real number, and $x^\alpha$ exists. The $\alpha$-fractional exponential function is defined by

\[
E_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(k+1))} x^{\alpha k} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha+1)} (1) x^{\alpha}^{\alpha k}.
\]

(18)

Remark 2.10: The smallest positive real number $T_{\alpha}$ such that $E_{\alpha}(iT_{\alpha}) = 1$, is called the period of $E_{\alpha}(ix^\alpha)$.

Definition 2.11([16]): The $\alpha$-fractional cosine and sine function are defined respectively as follows:

\[
\cos_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2\alpha k+1)} x^{2\alpha k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left( \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{2\alpha k},
\]

(19)

\[
\sin_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma((2k+1)\alpha+1)} x^{(2k+1)\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left( \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{(2k+1)}.
\]

(20)

In addition,

\[
\sec_{\alpha}(x^\alpha) = \left( \cos_{\alpha}(x^\alpha) \right)^{-1}
\]

(21)

is called the $\alpha$-fractional secant function.

\[
\csc_{\alpha}(x^\alpha) = \left( \sin_{\alpha}(x^\alpha) \right)^{-1}
\]

(22)

is the $\alpha$-fractional cosecant function.

\[
\tan_{\alpha}(x^\alpha) = \sin_{\alpha}(x^\alpha) \otimes \sec_{\alpha}(x^\alpha)
\]

(23)

is the $\alpha$-fractional tangent function. And

\[
\cot_{\alpha}(x^\alpha) = \cos_{\alpha}(x^\alpha) \otimes \csc_{\alpha}(x^\alpha)
\]

(24)

is the $\alpha$-fractional cotangent function.

Next, we introduce inverse fractional trigonometric functions.

Definition 2.12([17]): Suppose that $0 < \alpha \leq 1$. Then $arcsin_{\alpha}(x^\alpha)$ is the inverse function of $\sin_{\alpha}(x^\alpha)$, and it is called inverse $\alpha$-fractional sine function. $arccos_{\alpha}(x^\alpha)$ is the inverse function of $\cos_{\alpha}(x^\alpha)$, and we say that it is the inverse $\alpha$-fractional cosine function.
fractional cosine function. On the other hand, \( \text{arctan}_a(x^a) \) is the inverse function of \( \tan_a(x^a) \), and it is called the inverse \( \alpha \)-fractional tangent function. \( \text{arccotan}_a(x^a) \) is the inverse function of \( \cot_a(x^a) \), and it is the inverse \( \alpha \)-fractional cotangent function. \( \text{arcsec}_a(x^a) \) is the inverse function of \( \sec_a(x^a) \), and it is the inverse \( \alpha \)-fractional secant function. \( \text{arccsc}_a(x^a) \) is the inverse function of \( \csc_a(x^a) \), and it is called the inverse \( \alpha \)-fractional cosecant function.

The main methods used in this paper are introduced below.

**Theorem 2.13** ([17]): Suppose that \( 0 < \alpha \leq 1 \) and \( \left| \frac{1}{\Gamma(a+1)} x^a \right| < 1 \). Then

\[
(\mathcal{D}_x^a)[\arcsin_a(x^a)] = \left[ 1 - \left( \frac{1}{\Gamma(a+1)} x^a \right)^{\Theta(\frac{1}{2})} \right], \tag{25}
\]

\[
(\mathcal{D}_x^a)[\arccos_a(x^a)] = - \left[ 1 - \left( \frac{1}{\Gamma(a+1)} x^a \right)^{\Theta(\frac{1}{2})} \right], \tag{26}
\]

\[
(\mathcal{D}_x^a)[\arctan_a(x^a)] = \left[ 1 + \left( \frac{1}{\Gamma(a+1)} x^a \right)^{\Theta(\frac{1}{2})} \right], \tag{27}
\]

\[
(\mathcal{D}_x^a)[\text{arccot}_a(x^a)] = - \left[ 1 + \left( \frac{1}{\Gamma(a+1)} x^a \right)^{\Theta(\frac{1}{2})} \right]. \tag{28}
\]

**Theorem 2.14** ([17]): Let \( 0 < \alpha \leq 1 \), then

\[
\arcsin_a(x^a) + \arccos_a(x^a) = \frac{T_\alpha}{4}, \tag{29}
\]

and

\[
\arctan_a(x^a) + \text{arccot}_a(x^a) = \frac{T_\alpha}{4}. \tag{30}
\]

**Theorem 2.15** ([14]): If \( 0 < \alpha \leq 1 \) and \( r \) is a real number, then the \( \alpha \)-fractional binomial series

\[
\left(1 + \frac{1}{\Gamma(a+1)} x^a \right)^{\Theta(r)} = \sum_{k=0}^\infty \frac{(-1)^k (r)_k}{k!} x^k \Gamma(k+1), \tag{31}
\]

and

\[
\left(1 - \frac{1}{\Gamma(a+1)} x^a \right)^{\Theta(r)} = \sum_{k=0}^\infty \frac{(-1)^k (r)_k}{k!} x^k \Gamma(k+1). \tag{32}
\]

Where \( (r)_k = r(r - 1) \cdots (r - k + 1) \) for any positive integer \( k \), \( (r)_0 = 1 \), and \( \left| \frac{1}{\Gamma(a+1)} x^a \right| < 1 \).

### III. MAIN RESULTS

**Theorem 3.1**: If \( 0 < \alpha \leq 1 \), \( x \) is a real number, and \( \left| \frac{2}{\Gamma(2\alpha+1)} x^{2\alpha} \right| < 1 \). Then the \( \alpha \)-fractional Taylor series of some inverse \( \alpha \)-fractional trigonometric functions

\[
\arcsin_a(x^a) = \sum_{k=0}^\infty \frac{[(2k)!]^2}{2k!(k!)^2 \Gamma((2k+1)a+1)} x^{(2k+1)a}, \tag{33}
\]

\[
\arccos_a(x^a) = \frac{T_\alpha}{4} - \sum_{k=0}^\infty \frac{[(2k)!]^2}{2k!(k!)^2 \Gamma((2k+1)a+1)} x^{(2k+1)a}, \tag{34}
\]

\[
\arctan_a(x^a) = \sum_{k=0}^\infty \frac{(-1)^k (2k)!}{k! \Gamma((2k+1)a+1)} x^{(2k+1)a}, \tag{35}
\]

\[
\text{arccot}_a(x^a) = \frac{T_\alpha}{4} - \sum_{k=0}^\infty \frac{(-1)^k (2k)!}{k! \Gamma((2k+1)a+1)} x^{(2k+1)a}. \tag{36}
\]
Proof: \( \arcsin_a(x^a) = \left( \partial_{x}^{a} \right) \left[ \left( 1 - \left( \frac{1}{\Gamma(a) + 1} \right)^{\alpha} \right)^{2} \right] \) (by Theorem 2.13)

\[
= \left( \partial_{x}^{a} \right) \left[ \sum_{k=0}^{\infty} \frac{(-1)^{k} \binom{-1}{k} \Gamma(a + 1)}{k!} x^{a} \right] \Theta(2k) \quad \text{(by Theorem 2.15)}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{k} \Gamma(a + 1)}{k!} \left( \partial_{x}^{a} \right) \left[ x^{a} \right] \Theta(2k) \quad \text{(by Theorem 2.15)}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{k} \Gamma(a + 1)}{k!(2k+1)^{4}} x^{(2k+1)a} \quad .
\]

Moreover, since \( \arcsin_a(x^a) + \arccos_a(x^a) = \frac{\pi}{a} \), it follows that

\[
\arccos_a(x^a) = \frac{\pi}{a} - \sum_{k=0}^{\infty} \frac{\binom{2k}{2}^{2}}{2^{2k}(k!)^{2}} \Gamma((2k+1)a+1) x^{(2k+1)a} .
\]

On the other hand,

\[
\left( \partial_{x}^{a} \right) \left[ \arctan_a(x^a) \right] = \left[ 1 + \left( \frac{1}{\Gamma(a+1)} \right)^{\alpha} \right] \Theta(2) \quad \text{(by Theorem 2.13)}
\]

\[
= \left( \partial_{x}^{a} \right) \left[ \sum_{k=0}^{\infty} \frac{(-1)^{k} \binom{-1}{k} \Gamma(a + 1)}{k!} x^{a} \right] \Theta(2k) \quad \text{(by Theorem 2.15)}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{k} \Gamma(a + 1)}{k!} \left( \partial_{x}^{a} \right) \left[ x^{a} \right] \Theta(2k) \quad \text{(by Theorem 2.15)}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{k} \Gamma(a + 1)}{2^{2k+1} \Gamma((2k+1)a+1)} x^{(2k+1)a} \quad .
\]

Furthermore, since \( x^a + \arccot_a(x^a) = \frac{\pi}{a} \), it follows that

\[
\arccot_a(x^a) = \frac{\pi}{a} - \sum_{k=0}^{\infty} \frac{(-1)^{k} \binom{2k}{2}^{2}}{2^{2k}(k!)^{2}} \Gamma((2k+1)a+1) x^{(2k+1)a} \quad .
\]

Q.e.d.

Notation 3.2: If \( s \) is a real number, then the smallest integer greater than or equal to \( s \) is denoted as \( \lceil s \rceil \).

Theorem 3.3: Assume that \( 0 < \alpha \leq 1 \), \( n \) is a positive integer, \( x \) is a real number, and \( \left| \frac{2}{\Gamma(2a+1)} x^{2a} \right| < 1 \). Then the \( n- \) th order fractional derivative of inverse \( \alpha \)-fractional trigonometric functions are

\[
\left( \partial_{x}^{a} \right)^{n} \arcsin_a(x^a) = \sum_{k=0}^{\infty} \frac{\binom{2k}{2}^{2}}{2^{2k}(k!)^{2}} \Gamma((2k-n+1)a+1) x^{(2k-n+1)a} .
\]

(37)

\[
\left( \partial_{x}^{a} \right)^{n} \arccos_a(x^a) = - \sum_{k=0}^{\infty} \frac{\binom{2k}{2}^{2}}{2^{2k}(k!)^{2}} \Gamma((2k-n+1)a+1) x^{(2k-n+1)a} .
\]

(38)
\[(\partial D_x^n)^[\tan_a(x^a)] = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{\Gamma((2k-n+1)\alpha+1)} x^{(2k-n+1)\alpha}, \quad (39)\]

\[(\partial D_x^n)^[\cot_a(x^a)] = -\sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{\Gamma((2k-n+1)\alpha+1)} x^{(2k-n+1)\alpha}. \quad (40)\]

**Proof**

\[(\partial D_x^n)^[\sin_a(x^a)] = (\partial D_x^n)^[\tan_a(x^a)] \frac{T_x}{4} - \frac{\partial D_x^n}{\tan_a(x^a)}\]

\[(\partial D_x^n)^[\cos_a(x^a)] = \left(-\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2 \Gamma((2k-n+1)\alpha+1)} x^{(2k-n+1)\alpha}\right) \frac{T_x}{4} - \frac{\partial D_x^n}{\tan_a(x^a)}\]

\[(\partial D_x^n)^[\tan_a(x^a)] = \left(\partial D_x^n\right)^[\sin_a(x^a)] \frac{T_x}{4} - \frac{\partial D_x^n}{\tan_a(x^a)}\]

\[(\partial D_x^n)^[\cot_a(x^a)] = \left(-\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2 \Gamma((2k-n+1)\alpha+1)} x^{(2k-n+1)\alpha}\right) \frac{T_x}{4} - \frac{\partial D_x^n}{\tan_a(x^a)}\]

Q.E.D.

**IV. CONCLUSION**

As mentioned above, this paper obtained any order fractional derivative of inverse fractional trigonometric function based on Jumarie type of fractional derivative. The main methods we used are the fractional binomial series and a new multiplication of fractional analytic functions. In fact, these results in this study are generalizations of those in classical calculus. Moreover, the new multiplication is a natural operation of fractional analytic functions. In the future, we will continue to study the problems in applied mathematics and fractional calculus by using the new multiplication and the fractional binomial series.

**REFERENCES**


