

Bessel Functions and Their Applications: Solution to Schrödinger equation in a cylindrical function of the second kind and Hankel Functions

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Abstract: Bessel functions are a series of solutions to a second order differential equation that arise in many diverse situations, Develops the different kinds of Bessel functions and explores the topic of zeroes. In a solution of a free particle (zero potential) time-independent Schrodinger equation as applied to cylindrical function of the second kind (Neumann functions) and cylindrical function of third kind (Hankel functions of first and second kind). we would see that the solution in each case which is present in the solution of Bessel differential equation are the same.

Keywords: Bessel Equation, Bessel Functions, Modified Bessel Equation, Modified Bessel Functions, Neumann Function, Hankel Functions, Schrodinger Equation.

1. INTRODUCTION

Bessel function were studied by Euler, Lagrange and the Bernoulli. The Bessel functions were first used by Friedrich Wilhelm Bessel to explain the three body motion, with the Bessel function which emerge in the series expansion of planetary perturbation. Bessel function are named for Friedrich Wilhelm Bessel (1784-1846), after all, Daniel Bernoulli is generally attributed with being the first to present the idea of Bessel functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain hanging at one end. By the year 1764, Leonhard Euler employed Bessel functions of both the integral orders and zero orders in an analysis of vibrations of a stretched membrane, a research that was further developed(refined) by Lord Rayleigh in 1878, where he proved that Bessel functions are particular case of Laplace functions (Niedziela,2008). The Bessel function was the result of Bessel study of a problem of Kepler for determining the motion of three bodies moving under mutual gravitation. In 1824, he incorporated Bessel functions in a study of planetary perturbations where the Bessel functions appear as coefficients in a series expansion of the indirect perturbation of a planet, that is the motion of the Sun caused by the perturbing body.

Bessel's differential equation arises as a result of determining separable solutions to Laplace's equation and the Helmholtz equation in spherical and cylindrical coordinates. Therefore, Bessel functions are of great important for many problems of wave propagation and static potentials.

In 1926, Erwin Schrödinger as a result of his interest by the de Broglie hypothesis created an equation as a way of expressing the wave behavior of matter particle, for example, the electron. The equation was later named as Schrödinger equation which can be written as:

$$\left(\frac{-\hbar}{2m} \nabla^2 + U(r, t) \right) \psi(r, t) = i\hbar \psi(r, t)$$

Where m is the mass of the particle, $U(r, t)$ is the potential energy, ∇^2 is the Laplacian, and $\psi(r, t)$ is the wave function. Indeed, the Schrödinger equation given above is of most important and fundamental equation of the modern physics, the time dependent Schrödinger equation for a quantum system is introduced as a powerful analog of Newton's second law of motion for a classical system. However, we consider only the time independent Schrödinger equation for a free particle (Griffiths, 1995).

2. BESSEL'S DIFFERENTIAL EQUATION

Bessel's equation and Bessel's function occurs in relation with many problems of engineering and physics also there is an extensive literature that deals with the theory and application of this equation and its solution.

Bessel's equation can be used to find a solution of Laplace's equation (that is the key equation in the field of mathematical physics) related with the circular cylinder functions.

In Cartesian coordinates, the Laplace's equation is given by:

$$\nabla^2 K = \frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} + \frac{\partial^2 K}{\partial z^2} = 0 \quad (1)$$

Where ∇^2 is the Laplacian operator. Now we are more concerned in finding the solution of Laplace's equation using cylindrical coordinates. In such a coordinate system the equation can be written as follows:

$$\frac{1}{q} \frac{\partial}{\partial q} \left(q \frac{\partial K}{\partial q} \right) + \frac{1}{q^2} \frac{\partial^2 K}{\partial h^2} + \frac{\partial^2 K}{\partial z^2} = 0$$

Implies;

$$\frac{\partial^2 K}{\partial q^2} + \frac{1}{q} \frac{\partial K}{\partial q} + \frac{1}{q^2} \frac{\partial^2 K}{\partial h^2} + \frac{\partial^2 K}{\partial z^2} = 0 \quad (2)$$

We use separation of variables method to solve this equation, which is a method used to solve many kind of partial differential equations.

3. DEFINITIONS

1. Bessel Equation

The second order differential equation given as

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

is known as Bessel's equation. Where the solution to Bessel's equation yields Bessel functions of the first and second kind as follows:

$$Y = A J_\nu(x) + B Y_\nu(x) \quad (3)$$

where A and B are arbitrary constants. While Bessel functions are often presented in text books and tables in the form of integer order, i.e. $\nu = 0, 1, 2, \dots$, in fact they are defined for all real values of $-\infty < \nu < \infty$.

2. Bessel Functions

a) First Kind: $J_\nu(x)$ in the solution to Bessel's equation is referred to as a Bessel function of the first kind.

b) Second Kind: $Y_\nu(x)$ in the solution to Bessel's equation is referred to as a Bessel function of the second kind or sometimes the Weber function or the Neumann function.

c) Third Kind: The Hankel function or Bessel function of the third kind can be written as

$$H_\nu^1(x) = J_\nu(x) + iY_\nu(x) \quad x > 0$$

$$H_\nu^2(x) = J_\nu(x) - iY_\nu(x) \quad x < 0$$

Because of the linear independence of the Bessel function of the first and second kind, the Hankel functions provide an alternative pair of solutions to the Bessel differential equation.

3. Modified Bessel Equation

By letting $x = ix$ (where $i = \sqrt{-1}$) in the Bessel equation we can obtain the modified Bessel equation of order ν , given as

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2)y = 0$$

The solution to the modified Bessel equation yields modified Bessel functions of the first and second kind as follows:

$$Y = CL_\nu(x) + DK_\nu(x) \quad x > 0, \quad (4)$$

4. Modified Bessel Functions

- a) **First Kind:** $L_\nu(x)$ in the solution to the modified Bessel's equation is referred to as a modified Bessel function of the first kind.
- b) **Second Kind:** $K_\nu(x)$ in the solution to the modified Bessel's equation is referred to as a modified Bessel function of the second kind or sometimes the Weber function or the Neumann function.

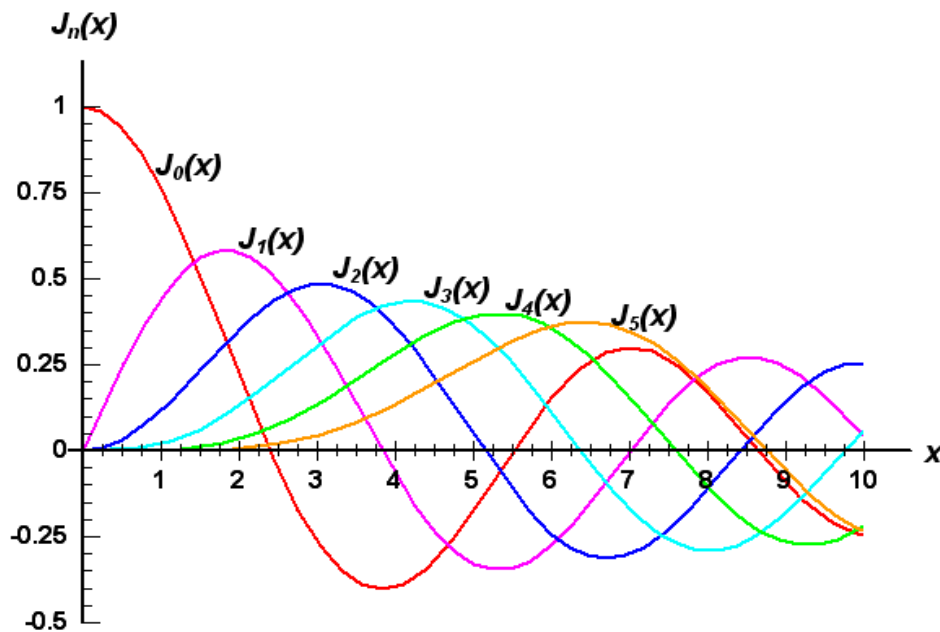


Figure 1: Bessel function of the first kind

Bessel function of the first kind for m equal to semi-integers

The confinement can be considered less strong when we prove that J_ν and $J_{-\nu}$ are independent when ν is equal to semi-integer. For such values of ν , the equation can be expressed as a combination of algebraic and trigonometric functions.

Now, consider $J_{\frac{1}{2}}$, from equation (2.18) we obtained:

$$J_{\frac{1}{2}}(x) = \sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+1/2} n! \Gamma(n+3/2)} = \frac{\sqrt{2}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n! \Gamma(n+3/2)} \quad (5)$$

To obtain a solution of the above equation we need to simplify the denominator. Firstly, the gamma function can be written as:

$$\Gamma(n + 3/2) = (n + 1/2) \cdot (n - 1/2) \cdot \dots \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)$$

Since

$$\Gamma(1/2) = \sqrt{\pi}$$

Thus:

$$\Gamma(n + 3/2) = \frac{1}{2^{n+1}} (2n + 1) \cdot (2n - 1) \cdot \dots \cdot 3 \cdot 1 \cdot \sqrt{\pi} \tag{6}$$

From the denominator of equation (2.25) we can also have:

$$2^{2n+1} n! = 2 \cdot 2^n \cdot 2^n \cdot n \cdot (n - 1) \cdot \dots \cdot 2 \cdot 1 = 2^{2n+1} \cdot (2n) \cdot (2n - 2) \cdot \dots \cdot 4 \cdot 2 \tag{7}$$

By putting equation (5) and (6) into (7), we obtained:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!} = \sqrt{\frac{2}{\pi x}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

The expression inside the bracket is McLaurin Series for $\sin(x)$. Thus, we have:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \tag{8}$$

Similarly,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \tag{9}$$

From equation (8) and (9) we see that $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$ are independent functions. Also, by using the recurrence relations, we can find the Bessel function for any index of the form $n + 1/2$, where n is an integer, and prove that for all integer n the following formulae holds:

$$J_{n+\frac{1}{2}}(x) = \frac{(-1)^n (2x)^{n+\frac{1}{2}}}{\sqrt{\pi}} \frac{d^n}{(dx^2)^n} \left(\frac{\sin(x)}{x} \right) \tag{10}$$

$$J_{-n+\frac{1}{2}}(x) = \frac{(-1)^n (2x)^{n+\frac{1}{2}}}{\sqrt{\pi}} \frac{d^n}{(dx^2)^n} \left(\frac{\cos(x)}{x} \right) \tag{11}$$

For the modified Bessel function, we use the same method and we can have:

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

And

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} e^{-x}$$

We can also use

$$Q(x) = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x)$$

As the general solution for Bessel's differential equation with $v = 1/2$. Without a doubt, all Bessel function with v equal to a half-integer, could be expressed in terms of elementary algebraic and trigonometric functions, and for these values of

v , J_v will always be independent of J_{-v} . Sometimes we can call Bessel functions for semi-integer values of v as spherical Bessel functions. Thus, we can re-write the general solution (2.20), as:

$$Q(x) = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x), \quad v \geq 0, \neq k, k = 0, 1, 2, 3, \dots \quad (12)$$

And it converges for all real $x > 0$.

Zeroes of Bessel Functions

The zeroes of Bessel functions are of great importance in applications [5]. The zeroes, or roots, of the Bessel functions are the values of x where value of the Bessel function goes to zero ($J_v(x) = 0$). Frequently, the zeroes are found in tabulated formats, as they must be numerically evaluated [5]. Bessel functions of the first

FIG. 2: The Neumann Functions (black) and the Modified Bessel Functions (blue) for integer orders $v = 0$ to $v = 5$

FIG. 3: The Modified Bessel Functions of the second kind for orders $v = 0$ to $v = 5$ [4]

and second kind have an infinite number of zeros as the value of x goes to ∞ . The zeroes of the functions can be seen in the crossing points of the graphs in figure (1), and figure (2). The modified Bessel functions of the first kind ($I_v(x)$) have only one zero at the point $x = 0$, and the modified Bessel equations of the second kind ($K_v(x)$) functions do not have zeroes.

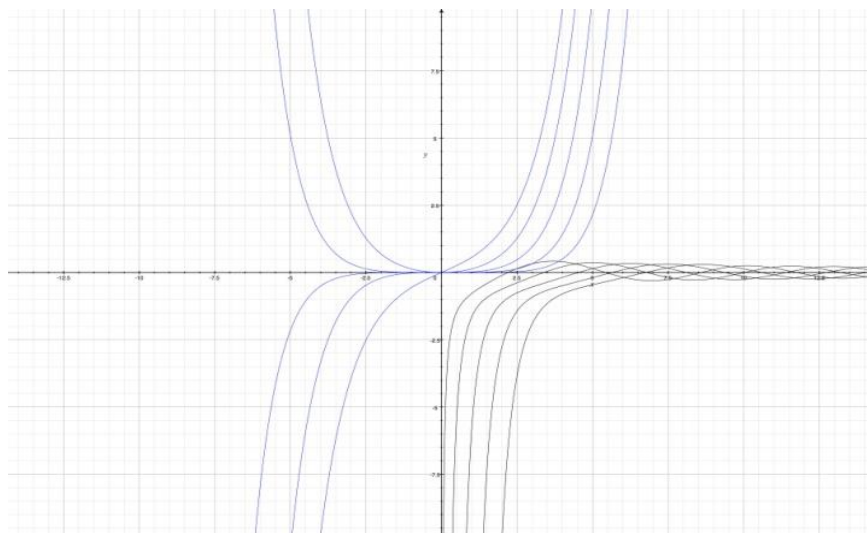


FIG. 2: The Neumann Functions (black) and the Modified Bessel Functions (blue) for integer orders $v = 0$ to $v = 5$

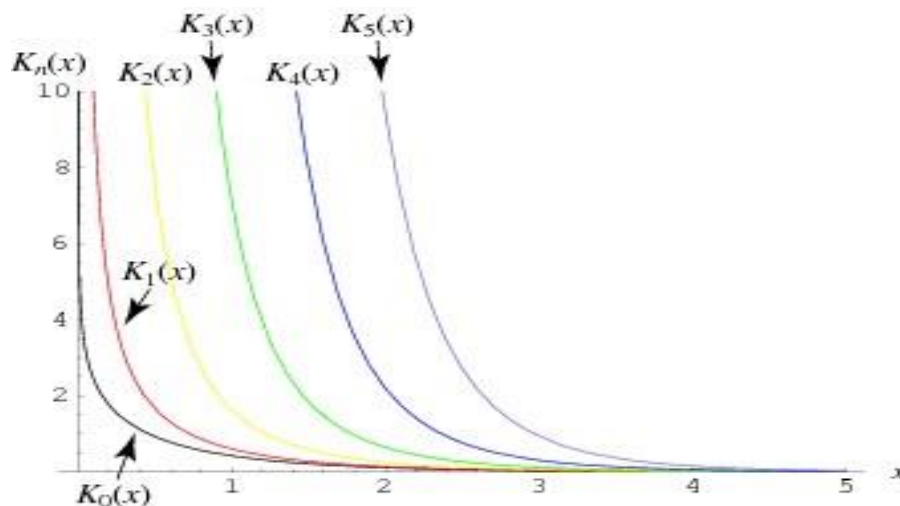


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Modified Bessel function (Cylindrical Functions of a Pure Imaginary Arguments)

Modified Bessel functions are solutions of the modified Bessel’s differential equation.

Now, consider the Bessel’s differential equation:

$$\frac{1}{x} \frac{d}{dx} \left(x \frac{dQ}{dx} \right) - \left(1 + \frac{v^2}{x^2} \right) Q = 0 \tag{13}$$

This equation will shows up if we make a simple transformation $x \rightarrow ix$ because we have to observe not only asymptotic at $x \rightarrow 0$, but also asymptotic at $x \rightarrow \infty$.

$$(ix)^2 \left(\frac{-d^2Q}{dx^2} \right) + \frac{(ix)}{i} \left(\frac{dQ}{dx} \right) + ((ix)^2 - v^2)Q = 0$$

Implies:

$$-x^2 \left(\frac{-d^2Q}{dx^2} \right) + x \left(\frac{dQ}{dx} \right) + (-x^2 - v^2)Q = 0$$

$$x^2 \left(\frac{d^2Q}{dx^2} \right) + x \left(\frac{dQ}{dx} \right) - (x^2 + v^2)Q = 0$$

$$x^2 Q'' + x Q' - (x^2 + v^2)Q = 0$$

Which is called the modified Bessel function. And has a regular singular point at $x = 0$. We also use Frobenius method to obtain a solution of Modified Bessel function. One solution $I_\nu(x)$ of equation (2.33) is defined by the series

$$I_\nu(x) = \left(\frac{x}{2} \right)^v \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+\nu+1)} \cdot \left(\frac{x}{2} \right)^{2n} \tag{14}$$

At $x \rightarrow 0$

$$I_\nu \approx \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \tag{15}$$

At $x \rightarrow \infty$

$$I_\nu(x) = \sqrt{\frac{2}{\pi x}} e^x \tag{16}$$

$I_\nu(x)$ is the real function of real argument? They are related with Bessel functions of the first kind by:

$$I_\nu(x) = e^{-\frac{\pi}{2} \nu i} J_\nu(ix) \tag{17}$$

In particular,

$$I_m(x) = -i^m J_m(ix) \tag{18}$$

Modified Bessel functions of second kind are defined by the relation

$$K_\nu(x) = \frac{\pi i}{2} e^{\frac{\pi}{2} \nu i} H_\nu^{(1)}(ix) \tag{19}$$

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} , \quad x \rightarrow \infty \tag{20}$$

They have asymptotic at both $I_\nu(x)$ and $K_\nu(x)$.

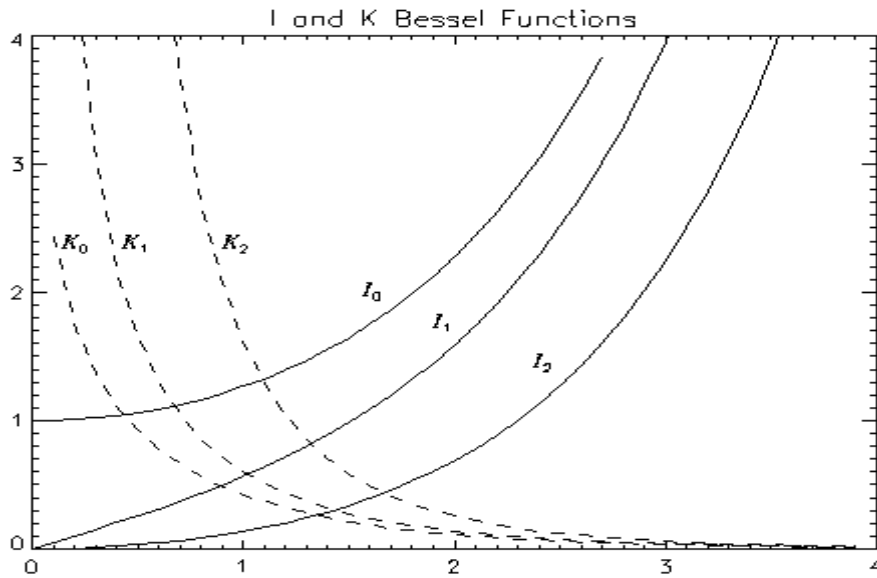


Figure 4: Modified Bessel function.

Cylindrical function of the second kind (Neumann or Weber's function)

At whatever point v is not an integer, a fundamental system for a solution of Bessel's differential equation for functions of order v is formed by a pair $J_v(x)$ and $J_{-v}(x)$. In case $v = m$ (m an integer), the functions $J_m(x)$ and $J_{-m}(x)$ are linearly dependent, so that $J_{-m}(x)$ is not a second solution of the equation. The second solution can be obtained as a combination of $J_v(x)$ and $J_{-v}(x)$ as follows:

$$Y_v(x) = \frac{J_v(x) \cos(\pi v) - J_{-v}(x)}{\sin(\pi v)} \tag{21}$$

This is weber's function (Neumann function) which satisfy Bessel's differential equation because it is linear combination of $J_v(x)$ and $J_{-v}(x)$. When $v = m$, the second solution is given by:

$$Y_m(x) = \lim_{v \rightarrow m} \frac{J_v(x) \cos(\pi v) - J_{-v}(x)}{\sin(\pi v)} \tag{22}$$

Also the general form of equation (2.42) above has been given by Neumann as:

$$Y_m(x) = J_m(x) \{ \log x - S_m \} - \sum_{n=0}^{m-1} \frac{2^{(m-n-1)} m! J_n(x)}{(m-n)! n! 2^{(m-n)}} + \sum_{n=0}^{m-1} \frac{(-1)^{(n-1)} (m+2n) J_{m+2n}(x)}{n(m+n)} \tag{23}$$

Where $S_m = 1 + 1/2 + 1/3 + \dots + 1/m$, $S_0 = 0$.

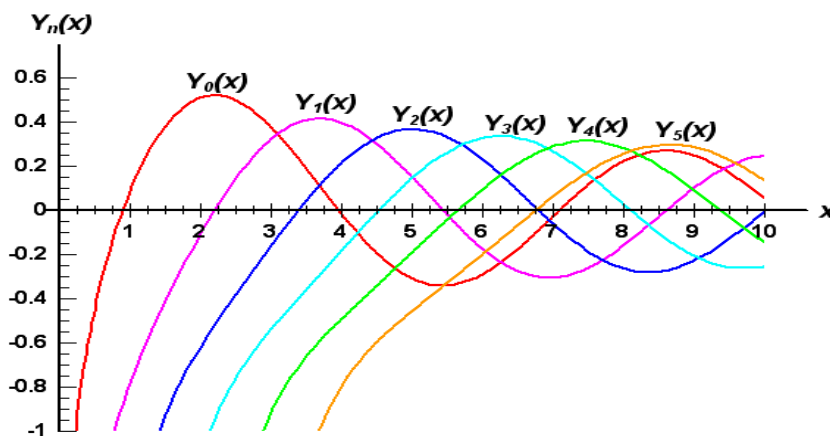


Figure 5: Bessel Function of the Second Kind.

Cylindrical function of the third kind (Hankel function)

Hankel function is a combination of Bessel's functions of the first kind ($J_v(x)$) and second kind ($Y_v(x)$). That is

$$H_v^{(1)}(x) = J_v(x) + jY_v(x) \tag{24}$$

$$H_v^{(2)}(x) = J_v(x) - jY_v(x)$$

Where $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$ represents Hankel functions of the first kind and second kind, respectively. Since the functions of the third kind, are linear combination of:

$$H_v^{(1)}(x) = J_v(x) + jY_v(x) = j \frac{e^{-v\pi j} J_v(x) - J_{-v}(x)}{\sin(v\pi)}$$

And

$$H_v^{(2)}(x) = J_v(x) - jY_v(x) = -j \frac{e^{v\pi j} J_v(x) - J_{-v}(x)}{\sin(v\pi)} \tag{25}$$

So that, as $x \rightarrow \infty$ they have the following asymptotic;

$$H_v^{(1)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} e^{j(x - \frac{\pi v}{2} - \frac{\pi}{4})}$$

And

$$H_v^{(2)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} e^{-j(x - \frac{\pi v}{2} - \frac{\pi}{4})}$$

Apparently,

$$H_v^{-(1)}(x) = H_v^{(2)}(x)$$

The above functions are linearly independent solutions of Bessel equations. Whereby v represents the degree of the Hankel functions of the first and second kind. When we add $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$ side by side, we obtained:

$$H_v^{(1)}(x) + H_v^{(2)}(x) = 2J_v(x)$$

$$J_v(x) = 1/2 [H_v^{(1)}(x) + H_v^{(2)}(x)] \tag{26}$$

Again, when we subtract the same equation, we can have,

$$H_v^{(1)}(x) - H_v^{(2)}(x) = 2jY_v(x)$$

$$Y_v(x) = 1/2j [H_v^{(1)}(x) - H_v^{(2)}(x)] \tag{27}$$

Therefore, the first and second kind Hankel functions are multiplied by $e^{jv\pi}$ and $e^{-jv\pi}$ respectively, and then adding them side by side, we obtained:

$$e^{jv\pi} H_v^{(1)}(x) + e^{-jv\pi} H_v^{(2)}(x) = 2J_{-v}(x)$$

$$J_{-v}(x) = 1/2 [e^{jv\pi} H_v^{(1)}(x) + e^{-jv\pi} H_v^{(2)}(x)] \tag{28}$$

Relations between the three kinds of Bessel functions

The relations express each of the function in terms of functions of other two kinds:

$$J_v(x) = \frac{H_v^{(1)}(x) + H_v^{(2)}(x)}{2} = \frac{Y_{-v}(x) + Y_v(x) \cos(\pi v)}{\sin(\pi v)} \quad (29)$$

$$J_{-v}(x) = \frac{e^{j\pi v H_v^{(1)}(x)} + e^{-j\pi v H_v^{(2)}(x)}}{2} = \frac{Y_{-v}(x) \cos(\pi v) - Y_v(x)}{\sin(\pi v)} \quad (30)$$

$$Y_v(x) = \frac{J_v(x) \cos(\pi v) - J_{-v}(x)}{\sin(\pi v)} = \frac{H_v^{(1)} - H_v^{(2)}(x)}{2j} \quad (31)$$

$$Y_{-v}(x) = \frac{J_{-v}(x) - J_v(x) \cos(\pi v)}{\sin(\pi v)} = \frac{e^{j\pi v H_v^{(1)}(x)} - e^{-j\pi v H_v^{(2)}(x)}}{2j} \quad (32)$$

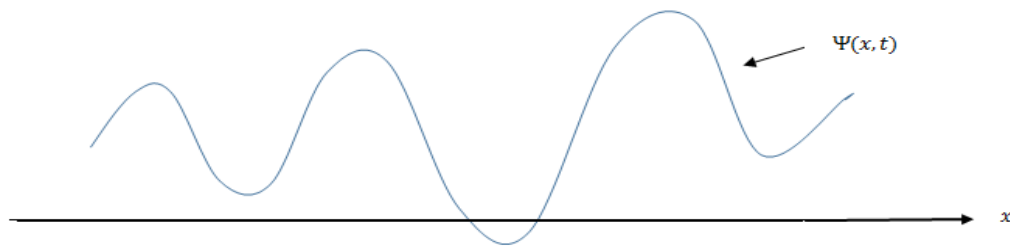
$$H_v^{(1)}(x) = \frac{J_{-v}(x) - e^{-j\pi v} J_v(x)}{j \sin(\pi v)} = \frac{Y_{-v}(x) - e^{-j\pi v} Y_v(x)}{\sin(\pi v)} \quad (33)$$

$$H_v^{(2)}(x) = \frac{e^{j\pi v} J_v(x) - J_{-v}(x)}{j \sin(\pi v)} = \frac{Y_{-v}(x) - e^{j\pi v} Y_v(x)}{\sin(\pi v)} \quad (34)$$

Derivation of time independent from the time dependent Schrodinger equation

The Schrödinger equation is the analog of the second law of motion and describes the motion and behavior of systems on the atomic and subatomic levels using the wave function $\Psi(x, t)$.

Consider a particle of mass m moving along the x-axis. At any position (x) and momentum (p) in time (t), the behavior and motion of the particle is given by the wave function $\Psi(x, t)$.



Where $\Psi(x, t)$ could take the form of any continues function that can be squared and integrated to get a finite answer. It is a fact from quantum mechanics that the wave function $\Psi(x, t)$ is a solution of the Schrödinger equation. (Tarasov, 2016).

The one dimensional time dependent Schrödinger equation is given by:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = u(x) \Psi(x, t) - \frac{\hbar^2}{2m} \cdot \frac{\partial^2 \Psi(x,t)}{\partial x^2} \quad (35)$$

From the above equation we can see that if we know what the wave function is at same initial time (say $t = 0$), we can use that to determine the behavior of that particle at some future time. i.e. if we know that $\Psi(x, 0)$ looks like, we can predict the future of the motion of the particle.

In quantum mechanics, we describe systems using wave functions. When we treat a system as a wave, the wave function represents the displacement of the wave. If we treat the system as a particle, then the wave function is use to give the probability of finding the particle at same point ($|\Psi|^2$). In order to describe any system in quantum mechanics, we must be able to determine what the wave function is numerically. The Schrödinger equation is a differential equation that we can use to solve for the wave function quantitatively. In the same way that Isaac Newton invented the second law of motion ($\sum f = \overline{ma}$). The Schrödinger equation was invented and confirmed using experiments. In order to determine what form the equation takes, we will use the conservation of energy. We will also assume that the wave function does not depend on time and only depends on the spatial position of the system $\Psi(x, t)$ (Griffiths, 1995).

Given;

$$\Psi(x, t) = A \sin(kx - \omega t) + B \cos(kx - \omega t)$$

Which is a classical wave equation. Since we are assuming time-independent Schrödinger equation.

Let $t = 0$,

Implies;

$$\Psi(x) = A \sin kx + B \cos kx \quad (36)$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{p}{\hbar/2\pi} = \frac{p}{\hbar}$$

Which implies,

$$k = \frac{p}{\hbar}$$

For a particle u , mass m and velocity v , the total energy is:

$$E = k + u = \frac{1}{2}mv^2 + u = \frac{p^2}{2m} + u$$

$$p = mv \Rightarrow x = \frac{p}{m}$$

$$\text{Since } p = k\hbar \Rightarrow E = \frac{k^2\hbar^2}{2m} + u \quad (37)$$

Since we are looking for a differential equation that look like equation (3.3) and has a solution that look like equation (3.2);

$$\Rightarrow \frac{d^2\Psi}{dx^2} = -k^2[A \sin(kx) + B \cos(kx)]$$

Which implies,

$$\frac{d^2\Psi}{dx^2} = -k^2\Psi(x) \quad (38)$$

Now, we multiply equation (3.4) by $\frac{-\hbar}{2m}$ and we obtain;

$$\frac{-\hbar}{2m} \frac{d^2\Psi}{dx^2} = \frac{\hbar^2 k^2}{2m} \Psi(x) \quad (39)$$

If we take equation (3.3) and multiply both sides by $\Psi(x)$, we have;

$$E \cdot \Psi(x) = \frac{\hbar^2 k^2}{2m} \Psi(x) + u \cdot \Psi(x)$$

Implies;

$$E \cdot \Psi(x) = u \cdot \Psi(x) - \frac{-\hbar}{2m} \cdot \frac{d^2\Psi}{dx^2} \quad (40)$$

The above equation is called one-dimensional time-independent Schrödinger equation.

Also, we can obtain the time-independent Schrödinger equation from the more general time-dependent equation using the method of separation of variable as follows:

Given the time independent Schrödinger equation:

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = u(x) - \frac{\hbar^2 k^2}{2m} \frac{d^2 \Psi(x, t)}{dx^2}$$

Let

$$\Psi(x, t) = \Psi(x) \cdot f(t) \quad (41)$$

We now substitute equation (3.7) into (3.1), we have:

$$i\hbar \frac{\partial \Psi(x)f(t)}{\partial t} = u(x)\Psi(x)f(t) - \frac{\hbar^2 k^2}{2m} \frac{d^2 \Psi(x)}{dx^2} \cdot f(t)$$

Dividing both sides of the above equation by $\Psi(x) \cdot f(t)$, we have;

$$i\hbar \frac{1}{f(t)} \cdot \frac{\partial f(t)}{\partial t} = u(x) - \frac{\hbar^2 k^2}{2m} \cdot \frac{1}{\Psi(x)} \cdot \frac{d^2 \Psi(x)}{dx^2} \tag{42}$$

So, the only way this equation can be equal is when both of them are equal to some constant, that is;

$$i\hbar \frac{1}{f(t)} \cdot \frac{\partial f(t)}{\partial t} = E \tag{43}$$

And

$$u(x) - \frac{\hbar^2 k^2}{2m} \cdot \frac{1}{\Psi(x)} \cdot \frac{d^2 \Psi(x)}{dx^2} = E \tag{44}$$

Now, let us take equation (3.10) and multiply both sides by $\Psi(x)$, we obtain;

$$u(x)\Psi(x) - \frac{\hbar^2 k^2}{2m} \cdot \frac{d^2 \Psi(x)}{dx^2} = E\Psi(x)$$

So that, for a free particle (zero potential) solutions, we have;

$$-\frac{\hbar^2 k^2}{2m} \cdot \frac{d^2 \Psi(x)}{dx^2} = E\Psi(x)$$

Solution to Schrödinger equation in a cylindrical function of the second kind (Neumann Functions)

Consider the functions J_ν and $J_{-\nu}$ which are two linearly independent solutions of the Bessel's equation as representatives of the Neumann or Weber's functions. That is,

$$Y_\nu(x) = N_\nu(x) = \frac{J_\nu(x) \cos \pi\nu - J_{-\nu}(x)}{\sin \pi\nu} \tag{45}$$

Which in the Schrödinger equation presents.

$$-\frac{\hbar^2 k^2}{2m} \cdot \frac{d^2 Y_\nu}{dx^2} = EY_\nu(x) \tag{46}$$

Now, we differentiate equation (45) for the second times and substitute into equation (46), as follows:

$$\begin{aligned} \frac{dY_\nu(x)}{dx} &= \frac{d}{dx} \left[\frac{J_\nu(x) \cos \pi\nu}{\sin \pi\nu} - \frac{J_{-\nu}(x)}{\sin \pi\nu} \right] \\ &= \frac{\cos \pi\nu}{\sin \pi\nu} J'_\nu(x) - \frac{1}{\sin \pi\nu} J'_{-\nu}(x) \end{aligned}$$

Again,

$$\frac{d^2 Y_\nu}{dx^2} = \frac{d}{dx} \left[\frac{\cos \pi\nu}{\sin \pi\nu} J'_\nu(x) - \frac{1}{\sin \pi\nu} J'_{-\nu}(x) \right]$$

Implies;

$$\frac{d^2 Y_\nu}{dx^2} = \frac{\cos \pi\nu}{\sin \pi\nu} J''_\nu(x) - \frac{1}{\sin \pi\nu} J''_{-\nu}(x) \tag{47}$$

Therefore, we substitute equation (3.13) into equation (3.12), we have;

$$-\frac{\hbar^2 k^2}{2m} \cdot \frac{d^2 Y_\nu}{dx^2} = EY_\nu(x)$$

Implies,

$$-\frac{\hbar^2}{2m} \left[\frac{\cos \pi v}{\sin \pi v} J_v''(x) - \frac{1}{\sin \pi v} J_{-v}''(x) \right] = E \left[\frac{J_v(x) \cos \pi v - J_{-v}(x)}{\sin \pi v} \right]$$

$$\frac{1}{\sin \pi v} [\cos \pi v J_v''(x) - J_{-v}''(x)] = \frac{-2mE}{\hbar^2} \left[\frac{1}{\sin \pi v} \right] [J_v(x) \cos \pi v - J_{-v}(x)]$$

$$\cos \pi v J_v''(x) - J_{-v}''(x) = \frac{-2mE}{\hbar^2} [J_v(x) \cos \pi v - J_{-v}(x)]$$

By letting $r^2 = \frac{2mE}{\hbar^2}$, we have:

$$\cos \pi v J_v''(x) - J_{-v}''(x) = -r^2 \cos \pi v J_v(x) + r^2 J_{-v}(x)$$

So that we can obtain;

$$\cos \pi v J_v''(x) + r^2 \cos \pi v J_v(x) = J_{-v}''(x) + r^2 J_{-v}(x)$$

And we can re-write the above equation as:

$$\cos \pi v [J_v''(x) + r^2 J_v(x)] = J_{-v}''(x) + r^2 J_{-v}(x) \tag{48}$$

Therefore, the only way this equation can be equal is when both of them is equal to some constant. That is;

$$J_{-v}''(x) + r^2 J_{-v}(x) = k \tag{49}$$

$$\cos \pi v [J_v''(x) + r^2 J_v(x)] = k \tag{50}$$

To simplify equation (3.15) and equation (3.16), we follow the method of undetermined coefficient and obtain the solution as follows;

$$J_{-v}(x)_{genaral} = J_{-v}(x)_{complementary} + J_{-v}(x)_{particular}$$

So, for equation (3.15), we have:

$$J_{-v}(x)_{complementary}: \quad J_{-v}''(x) + r^2 J_{-v}(x) = 0$$

The characteristics equation is $\lambda^2 + r^2 = 0$, which implies $\lambda = \pm ir$.

Therefore,

$$J_{-v}(x)_{complementary} = c_1 \cos rx + c_2 \sin rx \tag{51}$$

And for $J_{-v}(x)_{particular}$, we have:

$$\text{By letting } J_{-v}(x) = A_1 \Rightarrow J_{-v}'(x) = 0 \text{ and } J_{-v}''(x) = 0$$

$$\therefore J_{-v}(x) = A_1 \tag{52}$$

So, the general solution is now written as:

$$J_{-v}(x) = c_1 \cos rx + c_2 \sin rx + A_1$$

But, we know that $r = \frac{\sqrt{2mE}}{\hbar}$, therefore;

$$J_{-v}(x) = c_1 \cos \left(\frac{\sqrt{2mE}}{\hbar} x \right) + c_2 \sin \left(\frac{\sqrt{2mE}}{\hbar} x \right) + A_1 \tag{53}$$

Similarly, for equation (3.16), we have:

$$J_v(x) = c_3 \cos \left(\frac{\sqrt{2mE}}{\hbar} x \right) + c_4 \sin \left(\frac{\sqrt{2mE}}{\hbar} x \right) + A_2 \tag{54}$$

Remark:

Clearly, equation (3.19) and equation (3.20) are similar, this shows that $J_v(x)$ and $J_{-v}(x)$ are the two linearly independent solutions of the Bessel's differential equation which also appears in the Neumann (Weber's) functions.

Solutions to Schrödinger equation in a cylindrical function of the third kind (Hankel functions)

Here, also we are going to apply the Schrödinger equation to cylindrical functions of the third kind (Hankel functions) and obtain the solution of Bessel's differential equation. The Hankel function of the first and second kind are respectively given by:

$$H_v^{(1)}(x) = J_v(x) + iY_v(x) = i \frac{e^{-v\pi i} J_v(x) - J_{-v}(x)}{\sin \pi v} \tag{55}$$

And

$$H_v^{(2)}(x) = J_v(x) - iY_v(x) = -i \frac{e^{v\pi i} J_v(x) - J_{-v}(x)}{\sin \pi v} \tag{56}$$

Again, on applying equation (3.21) and equation (3.22) in the Schrödinger equation, that is;

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2 H_v^{(1)}(x)}{dx^2} = E H_v^{(1)}(x) \tag{57}$$

We obtain the solutions as follows:

Now, we differentiate equation (55) for the second time and substitute into equation (57), we have;

$$\begin{aligned} H_v^{(1)'}(x) &= \frac{d}{dx} \left(\frac{i e^{-v\pi i} J_v(x)}{\sin \pi v} \right) - \frac{d}{dx} \left(\frac{i J_{-v}(x)}{\sin \pi v} \right) \\ &= \frac{i e^{-v\pi i}}{\sin \pi v} \frac{d}{dx} J_v(x) - \frac{i}{\sin \pi v} \frac{d}{dx} J_{-v}(x) \end{aligned}$$

Which implies,

$$\begin{aligned} H_v^{(1)''}(x) &= \frac{d}{dx} \left(\frac{i e^{-v\pi i}}{\sin \pi v} \frac{d}{dx} J_v(x) \right) - \frac{d}{dx} \left(\frac{i}{\sin \pi v} \frac{d}{dx} J_{-v}(x) \right) \\ &= \frac{i e^{-v\pi i}}{\sin \pi v} \frac{d^2}{dx^2} J_v(x) - \frac{i}{\sin \pi v} \frac{d^2}{dx^2} J_{-v}(x) \\ &= \frac{i e^{-v\pi i}}{\sin \pi v} J_v''(x) - \frac{i}{\sin \pi v} J_{-v}''(x) \end{aligned} \tag{58}$$

By replacing equation (3.24) into equation (3.23), we have;

$$SS - \frac{\hbar^2}{2m} \left(\frac{i e^{-v\pi i}}{\sin \pi v} J_v''(x) - \frac{i}{\sin \pi v} J_{-v}''(x) \right) = E \left(\frac{i e^{-v\pi i}}{\sin \pi v} J_v(x) - \frac{i}{\sin \pi v} J_{-v}(x) \right)$$

Letting $a^2 = \frac{2mE}{\hbar^2}$

Implies;

$$\frac{i}{\sin \pi v} J_{-v}''(x) - \frac{i e^{-v\pi i}}{\sin \pi v} J_v''(x) = a^2 \left[\frac{i e^{-v\pi i}}{\sin \pi v} J_v(x) - \frac{i}{\sin \pi v} J_{-v}(x) \right]$$

Implies;

$$\begin{aligned} \frac{i}{\sin \pi v} J_{-v}''(x) - \frac{i e^{-v\pi i}}{\sin \pi v} J_v''(x) &= \frac{i a^2 e^{-v\pi i}}{\sin \pi v} J_v(x) - \frac{i a^2}{\sin \pi v} J_{-v}(x) \\ \frac{i}{\sin \pi v} J_{-v}''(x) + \frac{i a^2}{\sin \pi v} J_{-v}(x) &= \frac{i e^{-v\pi i}}{\sin \pi v} J_v''(x) + \frac{i a^2 e^{-v\pi i}}{\sin \pi v} J_v(x) \end{aligned}$$

Therefore,

$$J''_{-v}(x) + a^2 J_{-v}(x) = e^{-v\pi i} (J''_v(x) + a^2 J_v(x)) \quad (59)$$

Again, the only way this equation can be equal is when both of them equal to some constant. That is:

Suppose;

$$J''_{-v}(x) + a^2 J_{-v}(x) = k_1 \quad (60)$$

And

$$e^{-v\pi i} (J''_v(x) + a^2 J_v(x)) = k_1 \quad (61)$$

So, we can obtain the solution as follows:

For equation (3.26), we have;

$$J_{-v}(x) = c_5 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_6 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_3 \quad (62)$$

And from equation (3.27), we have;

$$J_v(x) = c_7 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_8 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_4 \quad (63)$$

Remark:

Clearly, $J_v(x)$ and $J_{-v}(x)$ which are presents in equation (62) and equation (63) are linearly independent solutions of Bessel's differential equation that appears in the Hankel functions of the first kind.

Similarly, for the Hankel functions of the second kind in equation (57), we have the solution as follows:

$$J_{-v}(x) = c_9 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_{10} \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_5 \quad (64)$$

And

$$J_v(x) = c_{11} \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_{12} \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_6 \quad (65)$$

Lastly, the $J_v(x)$ and $J_{-v}(x)$ which are linearly independent solutions of Bessel's differential equation that appears in the Hankel functions of the second kind.

4. CONCLUSION

The Bessel functions appear in many diverse scenarios, particularly situations involving cylindrical symmetry. The most difficult aspect of working with the Bessel function is first determining that they can be applied through reduction of the system equation to Bessel's differential or modified equation, we have discussed the solution of a free particle (zero potential) time-independent Schrödinger equation as applied to cylindrical function of the second kind (Neumann functions) and cylindrical function of the third kind (Hankel functions of the first and second kind). It has been find out that, the solution in each case which are presents in the solution of Bessel differential equation are the same. The constants in each of the solution are to be determined via application of boundary conditions. This shows that the Bessel function appeared in many diverse scenarios, more especially in a situation involving cylindrical symmetry.

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