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Divisibility Test for Gaussian Integers

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Abstract: **Gaussian integers is a set of complex integers *a* + *bi* where both *a* and *b* are integers. Long division still the effective method to test the divisibility of a Gaussian integer by another Gaussian integer. In this paper, instead of using long division, we used the real and imaginary parts to test if a Gaussian integer is divisible by another Gaussian integer. General formulas are also developed in this study from which these tests may be derived.**

Keywords: **divisibility test, Gaussian integers.**

I. INTRODUCTION

Divisibility is introductory to the theory of numbers. The ancient Greeks explored topics like divisibility and tended to treat numbers with a near-mystical reverence, and attributed a great deal of importance and meanings to their findings [11]. Divisibility testing is an important concept required in application like cryptosystems and the most trivial way to perform divisibility testing is direct division, where the integer under test (dividend) is divided by the potential integer (divisor) to check if the remainder is zero [10].

Johann Carl Friedrich Gauss (1777-1855) and Leonhard Euler (1707-1783) both proposed extensions of integers that used irrational numbers that they were able to use to attempt to develop elegant proofs for Fermat's Last Theorem [6]. In 1995, Fermat's Last Theorem was proved by Andrew Wiles. A set of integers is extended by including additional values. The standard integers are the whole numbers together with the additive inverses of the natural numbers [12]. The extended integers include the standard integers and additional values such as i, $5i$ etc [8]. In other words, the extended integers include all numbers of the form $a + bi$ where a and b are real numbers and $i^2 = -1$. This set of extended integers forms the set of Gaussian integers, named after the Mathematician Johann Carl Friedrich Gauss**.** In this paper, we define the norm of a Gaussian integer as $N(a+bi)=(a+bi)(a-bi)$. We denote $Z[i]$ as the set of Gaussian integers.

The interesting idea behind Gaussian integers is that any theorem which is true for all Gaussian integers is also true for all integers since integers are Gaussian integers wherein $b = 0$. Before Gauss, extending the idea of fundamental theorem of Arithmetic for Gaussian integers was probably not interesting to Mathematicians because it is so clearly true in the case of integers. Gauss developed the properties of factorization for Gaussian integers and proved the uniqueness of the factorization of a Gaussian integer into prime Gaussian primes. The Euclidean Algorithm and Bezout's identity for Gaussian integers were developed because these were required in proving the unique factorization of complex integers [5,6,7,8].

II. MAIN RESULTS

2.1 Extensions of Divisibility Properties:

Table I shows the divisibility properties for integers with their corresponding properties extended in the set of Gaussian integers. The first column contained some divisibility properties for integers and the second column shows the extensions of divisibility properties of integers in the set of Gaussian integers.

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TABLE: I

Basic Properties for Divisibility of Integers	Extension in Gaussian Integers
1. For any integer a, $1\|a$ and $a\|a$.	1. For any Gaussian integer α, $1\|\alpha$ and $\alpha\|\alpha$.
2. If $a\|b$ and $b\neq 0$, then $\|a\|\leq\|b\|$.	2. If $\alpha \mid \beta$ and $\beta \neq 0$, then $\mid \alpha \mid \leq \mid \beta \mid$.
3. If $a \mid b$ and $b \mid c$, then $a \mid c$.	3. If $\alpha \mid \beta$ and $\beta \mid \gamma$, then $\alpha \mid \gamma$.
4. If $a \mid b$ and $a \neq 0$, then $\left(\frac{b}{a}\right) \mid b$.	4. If $\alpha \mid \beta$ and $\alpha \neq 0$, then $\left(\frac{\beta}{\alpha}\right) \mid \beta$.
5. If $ab \mid c$, then $a \mid c$.	5. If $\alpha \cdot \beta \mid \gamma$, then $\alpha \mid \gamma$.

2.2 Specific Divisibility Test for Gaussian Integers:

Let us start by presenting specific divisibility test for Gaussian integers. Suppose a Gaussian integer α is an integer that is the imaginary part of α is zero. To determine if α divides any Gaussian integer β is synonymous to verifying if α divides both the real and imaginary parts of β.

Now suppose α is a pure Gaussian integer, that is, none of its part is zero. It is easy to verify if $\alpha \nmid \beta$ (i.e. α does not divide β) means this may simply equal in verifying divisibility of their norm. In other words, if $N(\alpha) \nmid N(\beta)$, then $\alpha \nmid \beta$. Next suppose that α is a composite Gaussian integer whose norm is an even integer, then α is divisible by $1+i$. The presence of $1+i$ as a factor in an ordinary rational integer indicates evenness, and divisibility of a Gaussian integer α by $1+i$ is equivalent to divisibility of $N(\alpha)$ by 2 [4].

Theorem 2.2.1: Divisibility Test for 1 + *i*

If the norm of α is an even integers, then $(1 + i) \mid \alpha$.

This statement implies that any complex integer whose norm is even integer is a composite complex integer and is divisible by $1+i$. For instance, $(1+i) \mid (2+4i)$ since $N(2 + 4i) = 20$ and 20 is an even integer. The factors of $2+4i$ are $1+i$ and $3+i$. To explore and create a divisibility test for a particular Gaussian integer, by taking a clue from the Sieve of Erathostenes, Loy (1999) listed a few complex integers with norm less than 100 which are complex prime and this list included some prime integers. In this paper those prime integers were eliminated and only the pure Gaussian primes were considered. These numbers are $1+i$, $2+i$, $3+2i$, $4+i$, $5+2i$, $5+4i$, $7+2i$, $6+5i$, $8+3i$, $8+5i$, and $9+4i$. The divisibility test for the Gaussian integer $1+i$ has already been discussed. Divisibility tests for the remaining complex integers were explored through long division and were generalized and supplied with illustrations.

The list of prime complex integer of Loy (1999) has a restriction. He restricted the list to complex integers whose real part is greater than the coefficient of the imaginary part. In this paper, we included the Gaussian primes whose real part is less than the coefficient of the imaginary part in addition to the mentioned prime complex integers. Now the list of prime complex integers to be considered are $1+i$, $2+i$, $1+2i$, $3+2i$, $2+3i$, $4+i$, $1+4i$, $5+2i$, $2+5i$, $5+4i$, $4+5i$, $7+2i$, $2+7i$, $6+5i$, $5+6i$, $8+3i$, $3+8i$, $8+5i$, $5+8i$, $9+4i$, $4+9i$. The conjugate of these Gaussian primes are also prime but they are not included in the list for the reason that each is the product of one prime listed and a unit. For example, the Gaussian integer with norm 13 are $2+3i$, $-2+3i$, $2-3i$, $-2-3i$, $3+2i$, $-3+2i$, $3-2i$, and $-3-2i$, but whatever divisibility test applicable to $3+2i$ is also applicable to $2-3i$, $-3-2i$, and $-2+3i$, since $3+2i=i(2-3i)=(-1)(-3-2i)=(-i)(-2+3i)$. Similarly, whatever divisibility test applicable for the Gaussian integer $2+3i$ is also applicable to Gaussian integers $-2-3i$, $-3+2i$, and $3-2i$ since $2+3i=(-1)(-2-3i)=i(3-2i)=(-i)(-3+2i)$. The conjugate of $3+2i$ is $3-2i=i(2+3i)$. Therefore the researcher omitted those Gaussian primes whose parts are not both positive and developed divisibility tests only for Gaussian integers whose real and imaginary parts are positive. The theorem below generalizes the above example.

Theorem 2.2.2

Let $\zeta \in \{1,-1,i,-i\}$. If $\alpha \mid \beta$ then $\alpha\zeta \mid \beta$.

For example, if $(3+i) \mid (2+4i)$, then $(3+i)(-i) \mid (2+4i)$. The conclusion is true since $2+4i=(-1+i)(1-3i)=(-1+i)(3+i)(-i)$.

Let $\alpha=a+bi$ and $(2+i) \mid \alpha$. Now $(2+i) \mid \alpha$ is true if there exists a Gaussian integer say φ such that $\alpha=(2+i)\varphi$. Dividing both sides of this equation by $2+i$, yields

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$$\varphi = \frac{a+bi}{2+i} = \frac{(2a+b)+(2b-a)i}{5}$$

and φ is a Gaussian integer if $2a+b$ and $2b–a$ are both divisible by 5. Note that, 5 divides both $2a+b$ and $2b–a$ if b is replace by $3a$ or when a is replace by $2b$. The statement below generalizes the divisibility test and formalizes the concept.

Theorem 2.2.3: Divisibility Test for 2 + *i*

If $5 \mid (b\text{-}3a)$ or $5 \mid (a\text{-}2b)$, then $(2+i) \mid (a+bi)$.

Let us apply this test to verify divisibility. To verify if $(2+i) \mid (2+6i)$, take $a=2$ and $b=6$, and since $5 \mid (2–2(6))$, therefore $(2+i) \mid (2+6i)$. Now to determine the other factor of $2+6i$, recall that $2+6i = (2+i)[(2m+b)–mi]$. The value of m depends on $a–2b$, since $5m=a–2b=2–2(6)=–10=(–2)5$, so $m=–2$. This yields $(2m+b)–mi=[2(–2)+6]–(–2)i=2+2i$. Thus $2+6i=(2+i)(2+2i)$.

To derive a divisibility test for the Gaussian prime $1+2i$, observe that any divisibility test for the Gaussian prime $1+2i$ would be applicable to the Gaussian integer $2–i$ which is the complex conjugate of the Gaussian prime $2+i$. So the process will be the same by interchanging the real and coefficient of the imaginary part of the Gaussian integer to be tested. So the statement below formalizes the discussion.

Theorem 2.2.4: Divisibility Test for 1 + 2*i*

If $5 \mid (a–3b)$ or $5 \mid (b–2a)$, then $(1+2i) \mid (a+bi)$.

Proof. The statement $(1+2i) \mid (a+bi)$ means that there exists a Gaussian integer φ such that $a+bi=(1+2i)\varphi$. Dividing both sides of this equation by $1+2i$ yields $\frac{a+bi}{1+2i} = \varphi$. Now

$$\varphi = \frac{a+bi}{1+2i} = \frac{(a+2b)+(-2a+b)i}{5}$$

From the hypotheses

$5 \mid a – 3b \Longrightarrow a – 3b = 5n$ for some $n \in \mathbb{Z} \Longrightarrow a = 5n + 3b$

or

$5 \mid b – 2a \Longrightarrow b – 2a = 5m$ for some $m \in \mathbb{Z} \Longrightarrow b = 5m + 2a$

Thus

$$\varphi = \frac{a+bi}{1+2i} = \frac{(a+2b)+(-2a+b)i}{5} = \frac{[(5n+3b)+2b]+[-2(5n+3b)+b]i}{5}$$

$$= \frac{(5b+5n)+(-5b-10n)i}{5} = (b+n)+(-b-2n)i$$

or

$$\varphi = \frac{a+bi}{1+2i} = \frac{[a+2(5m+2a)]+[-2a+(5m+2a)]i}{5} = \frac{(5a+10m)+5mi}{5}$$

$$= (a+2m)+mi \,.\square$$

For example, let $\alpha=–1+18i$. To verify if α is divisible by $–2+i$. Divisibility test for $1+2i$ may be used since by Theorem 2.2.2, the divisibility test for $–2+i$ is the same as the divisibility test for $1+2i$ since $1+2i = i(–2+i)$. Let $a=–1$ and $b=18$, now $–1–3(18)=–1–54=–55$ and $5 \mid (–55)$. Hence $(–2+i) \mid (–1+18i)$.

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Observe that in the hypotheses of the divisibility tests for 2+*i* and 1+2*i*, the sum of the coefficients of the second terms of the dividends is equivalent to their divisor. Following this process the following divisibility tests were obtained.

Theorem 2.2.5: Divisibility Test for 3 + 2*i*

If $13 \mid (a-8b)$ or $13 \mid (b-5a)$, then $(3+2i) \mid (a+bi)$.

For example, for the Gaussian integer 8+14*i*, *a*=8 and *b*=14. Now $b-5a=14-5(8)=-26$ and $13 \mid (-26)$, therefore $(3+2i) \mid (8+14i)$.

2.3 Generalized Divisibility Tests

Since the divisor of the hypothesis of every divisibility tests is the norm of the particular Gaussian prime and equivalent to the sum of the second terms of the two dividends, so this procedure can be generalized by replacing the particular Gaussian prime. Now let $\beta=c+di$ be a particular Gaussian prime. In the preceding divisibility test, the hypothesis is given by $97 \mid (a-75b)$ or $97 \mid (b-22a)$. Note that 97=22+75, but this relationship has nothing to do with the Gaussian integer to be tested, it involves the integer parts of the divisor. For a particular divisor, say the Gaussian prime 9+4*i*, $N(9+4i)=97=22+75$ and $22=\frac{88}{4}=\frac{97-9}{4}$. Similarly $75=\frac{300}{4}=\frac{97(3)+9}{4}=\frac{97(4-1)+9}{4}$. Now let *c*=9 and *d*=4, so $N(c+di)=\frac{N(c+di)-c}{d}+\frac{N(c+di)\cdot(d-1)+c}{d}$.

The following theorems formalize this discussion and generalize the divisibility test for any Gaussian integer. The divisibility tests for the remaining Gaussian integers whose norms are prime can be derived using the following theorems.

Theorem 2.3.1

Let $\alpha=a+bi$ and $\beta=c+di$ where $c>d$. If $\frac{a(c-1)}{d}\in Z$ and $N(\beta) \mid b-\frac{N(\beta)-c}{d}\cdot a$ where $N(\beta)$ is prime in Z, then $\beta \mid \alpha$.

Proof. $\beta \mid \alpha$ if there exists a Gaussian integer φ such that $\alpha = \beta\varphi$. Since $\alpha=a+bi$ and $\beta=c+di$, then $a+bi=(c+di)\varphi$ and dividing both sides of this equation by *c di*, yields $\varphi=\frac{a+bi}{c+di}=\frac{(ac+bd)+(-ad+bc)i}{c^2+d^2}$. Next is to show that $\varphi\in Z[i]$. From the hypothesis

$$N(\beta) \mid b-\frac{N(\beta)-c}{d}\cdot a \Rightarrow b-\frac{N(\beta)-c}{d}\cdot a=N(\beta)\cdot n \text{ for some } n \text{ in Z}$$

$$\Rightarrow b=N(\beta)\cdot n+\frac{N(\beta)-c}{d}\cdot a.$$

$$\Rightarrow b=\frac{N(\beta)\cdot dn+N(\beta)\cdot a-ac}{d}$$

Now replacing the value of *b* yields

$$\varphi=\frac{\left\{ac+\left[\frac{N(\beta)\cdot dn+N(\beta)\cdot a-ac}{d}\right]\cdot d\right\}+\left\{-ad+\left[\frac{N(\beta)\cdot dn+N(\beta)\cdot a-ac}{d}\right]\cdot c\right\}i}{N(\beta)}$$

$$=\frac{\left[ac+N(\beta)\cdot dn+N(\beta)\cdot a-ac\right]+\left[\frac{-ad^2+N(\beta)\cdot cnd+N(\beta)\cdot ac-ac^2}{d}\right]i}{N(\beta)}$$

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$$= \frac{[N(\beta)\cdot nd + N(\beta)\cdot a] + \left[\frac{N(\beta)\cdot cnd + N(\beta)\cdot ac - a(c^2+d^2)}{d}\right]i}{N(\beta)}$$

$$= \frac{[N(\beta)\cdot nd + N(\beta)\cdot a] + \left[\frac{N(\beta)\cdot cnd + N(\beta)\cdot ac - N(\beta)\cdot a}{d}\right]i}{N(\beta)}$$

$$= (nd+a) + \left(\frac{cnd+ac-a}{d}\right)i$$

$$= (nd+a) + \left[cn + \frac{a(c-1)}{d}\right]i$$

Since $\frac{a(c-1)}{d} \in \mathbb{Z}$. Therefore $\left[cn + \frac{a(c-1)}{d}\right]$ is also an integer. Hence φ is a Gaussian integer. □

For example, to derive a divisibility test for 9+4*i*, let *c*=9 and *d*=4. First is to verify if $\frac{a(c-1)}{d}$ is an integer. Substituting *c* and *d* yields $\frac{a(9-1)}{4} = 2a$ which is an integer. Now *N*(9+4*i*)=97, so $b - \frac{97-9}{4}\cdot a$ = *b*–22*a*. Therefore the test can be stated as: If 97 | (*b* – 22*a*), then (9+4*i*) | (*a*+*bi*).

Theorem 2.3.2

Let *α*=*a*+*bi* and *β*=*c*+*di* where c>d. If $\frac{b(1-c)}{d} \in \mathbb{Z}$ and *N*(*β*) | $a - \frac{N(\beta)(d-1)+c}{d}\cdot b$ where *N*(*β*) is prime in Z, then *β* | *α*.

Proof. *β* | *α* if there exists a Gaussian integer *φ* such that *α*=*βφ*. Since *α*=*a*+*bi* and *β*=*c*+*di*, then *a*+*bi*=(*c*+*di*)*φ* and dividing both sides of this equation by *c*+*di*, yields $\varphi = \frac{a+bi}{c+di} = \frac{(ac+bd)+(-ad+bc)i}{c^2+d^2}$. Next is to show that $\varphi \in \mathbb{Z}[i]$. From the hypothesis

$$N(\beta) \mid a - \frac{N(\beta)(d-1)+c}{d}\cdot b \Rightarrow a - \frac{N(\beta)(d-1)+c}{d}\cdot b = N(\beta)\cdot n \text{ for some } n \in \mathbb{Z}$$

$$\Rightarrow a = N(\beta)\cdot n + \frac{(d-1)N(\beta)+c}{d}\cdot b$$

$$\Rightarrow a = \frac{N(\beta)\cdot dn + N(\beta)\cdot b(d-1) + bc}{d}$$

Now replacing the value of *a* yields

$$\varphi = \frac{\left\{\left[\frac{N(\beta)\cdot dn + N(\beta)\cdot b(d-1)+bc}{d}\right]\cdot c + bd\right\} + \left\{-\left[\frac{N(\beta)\cdot dn + N(\beta)\cdot b(d-1)+bc}{d}\right]\cdot d + bc\right\}i}{N(\beta)}$$

$$= \frac{\left[\frac{N(\beta)\cdot cdn + N(\beta)\cdot bc(d-1) + bc^2 + bd^2}{d}\right] + [-N(\beta)\cdot dn - N(\beta)\cdot b(d-1) - bc + bc]i}{N(\beta)}$$

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$$= \frac{\frac{N(\beta)\cdot cdn + N(\beta)\cdot bc(d-1) + b(c^2+d^2)}{d} + [-N(\beta)\cdot dn - N(\beta)\cdot b(d-1)]i}{N(\beta)}$$

$$= \frac{\frac{N(\beta)\cdot cdn + N(\beta)\cdot bc(d-1) + N(\beta)\cdot b}{d} - N(\beta)[dn + b(d-1)]i}{N(\beta)}$$

$$= \frac{\frac{N(\beta)[cdn + bc(d-1) + b]}{d} - N(\beta)[dn + b(d-1)]i}{N(\beta)}$$

$$= \frac{[cdn + bc(d-1) + b]}{d} - [dn + b(d-1)]i$$

$$= \left(cn + bc + \frac{b - bc}{d}\right) - [dn + b(d-1)]i$$

$$= \left[cn + bc + \frac{b(1-c)}{d}\right] + [b(1-d) - dn]i$$

Since $\frac{b(1-c)}{d} \in$ Z. Therefore $\left[cn + bc + \frac{b(1-c)}{d}\right]$ is also an integer. Hence φ is a Gaussian integer. □

For example, to derive a divisibility test for 7+2*i*, let *c*=7 and *d*=2. First is to verify if $\frac{b(1-c)}{d}$ is an integer. Substituting *c* and *d* yields $\frac{b(1-7)}{2} = -3b$ which is an integer. Now *N*(7+2*i*)=53, so $a - \frac{(2-1)53+7}{2}\cdot b$ =*a*–30b. Therefore the test can be stated as: If 53 | (*a*–30*b*), then (7+2*i*) | (*a*+*bi*).

Theorems 2.3.1 and 2.3.2 can only be used to derive a divisibility test for any Gaussian integer *c*+*di* when *c*>*d*. In deriving a divisibility test for Gaussian integer *c*+*di* where *c*<*d*, use the following two theorems. Observed, however, that any divisibility tests where *c*>*d* are also the same with the divisibility tests for *c*<*d* by interchanging the roles of the real and imaginary parts of the Gaussian integers to be tested. Theorem 2.3.3 and Theorem 2.3.4 are alternative formulas to derive a divisibility test for any Gaussian integer.

Theorem 2.3.3

Let $\alpha=a+bi$ and $\beta=c+di$. If $\frac{b(d-1)}{c} \in$ Z and $N(\beta) \mid a - \frac{N(\beta)-d}{c}\cdot b$ where $N(\beta)$ is prime in Z, then $\beta \mid \alpha$.

Proof. Similar to the proof of Theorem 2.3.1. □

For example, to derive a divisibility test for 2+5*i*, let *c*=2 and *d*=5. First is to verify if $\frac{b(d-1)}{c}$ is an integer. Substituting *c* and *d* yields $\frac{b(5-1)}{2} = 2b$ which is an integer. Now *N*(2+5*i*)=29, so $a - \frac{29-5}{2}\cdot b = a - 12b$. Therefore the test can be stated as: If 29 | (*a*–12*b*), then (2+5*i*) | (*a*+*bi*).

Theorem 2.3.4

Let $\alpha=a+bi$ and $\beta=c+di$. If $\frac{a(1-d)}{c} \in$ Z and $N(\beta) \mid b - \frac{N(\beta)(c-1)+d}{c}\cdot a$ where $N(\beta)$ is prime in Z, then $\beta \mid \alpha$.

Proof. Similar to the proof of Theorem 2.3.2. □

For example, to derive a divisibility test for 1+10*i*, let *c*=1 and *d*=10. First is to verify if $\frac{a(1-d)}{c}$ is an integer. Substituting *c* and *d* yields $\frac{a(1-10)}{1} = -9a$ which is an integer. Now *N*(1+10*i*)=101,

so $b - \frac{101(1-1)+10}{1} \cdot a = b - 10a$. Therefore the test can be stated as: If 101 | (*b*–10*a*), then (1+10*i*) | (*a*+*bi*). For instance, suppose *a*=22 and *b*=18. Because 101 | (18–10(22)), that is 101 | (–202), thus (1+10*i*) | (22+18*i*) and the factors of 22+18*i* are 1+10*i* and 2–2*i*.

III. CONCLUSION

Mathematics is a growing discipline. This subject is a source of many remarkable results particularly in Number Theory. Using simple algebraic manipulations, this exploration has shown how divisibility properties for integers may be extended into the set of Gaussian integers. The divisibility of two Gaussian integers may now be easily tested without using long division involving Gaussian integers. Some of the results of this study can also be used in factoring Gaussian integers.

It is verified that basic theorems on divisibility (found in most elementary number theory books) which are true for all integers are also true for all complex integers. Divisibility tests are derived based on divisibility theories for integers and the corresponding theories developed in this study. The coverage of number theory may be extended to include the study of Gaussian integers.

The set of Gaussian integers is a source of many interesting properties in number theory and it has been the subject of numerous investigations. The complex integers can be used to derive formulas such as the formulas in finding primitive Pythagorean triples, most of which do not appear in any elementary number theory book. The following are further studies recommended for investigation:

1. Divisibility properties and tests for Eisenstein integers and Kummer's complex integers;

2. Divisibility tests for Gaussian integers using the weighted sum;

3. Extensions in modular functions for Gaussian integers.

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