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Fractional Integral Problems of Some Fractional Trigonometric Functions

Chii-Huei Yu

School of Mathematics and Statistics,

Zhaoqing University, Guangdong, China

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***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study the fractional integral problems of fractional trigonometric functions. The solutions of the fractional integrals can be obtained by using some techniques. In addition, we give some examples to illustrate our results. On the other hand, our results are generalizations of the traditional calculus results.**

***Keywords*: Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, fractional integral, fractional trigonometric functions.**

I. INTRODUCTION

Fractional calculus is the study of derivatives and integrals of arbitrary orders. For a long time, the theory of fractional calculus developed only as a theoretical field of mathematics. However, in the last decades, it was shown that some fractional operators can better describe some complex physical phenomena, so fractional calculus has been paid more and more attention by mathematicians. On the other hand, physicists and engineers are also very interested in the applications of this nice theory. Many real life phenomena have been described using fractional differential equations, such as physics, viscoelasticty, mechanics, control theory, biology, electrical engineering, economics, and others [1-11].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study the following two α-fractional integrals of fractional trigonometric functions:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left[r^{2}cos_{\alpha}(x^{\alpha})+s^{2}sin_{\alpha}(x^{\alpha})\right]^{\otimes_{\alpha}(-1)}\right],$$

and

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left[r^{2}cos_{\alpha}(x^{\alpha})-s^{2}sin_{\alpha}(x^{\alpha})\right]^{\otimes_{\alpha}(-1)}\right].$$

Where $0<\alpha\leq 1$, r,s are real numbers and $r\neq 0, s\neq 0$. Using some methods, the solutions of these two fractional integrals can be obtained. In fact, our results are generalizations of ordinary calculus results.

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II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([17])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([18])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([21])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes_\alpha n}, \quad (9)$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes_\alpha n}. \quad (10)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \quad (11)$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \quad (12)$$

Definition 2.6 ([22])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (13)$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([23])**:** If $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \quad (14)$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (15)$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \quad (16)$$

Definition 2.8 ([24])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha (-1)}$.

III. MAIN RESULTS AND EXAMPLES

In this section, we find the solutions of two fractional integrals of fractional trigonometric functions. Moreover, we give some examples to illustrate our results.

Theorem 3.1: *If* $0 < \alpha \le 1$, r, s *are real numbers and* $r \ne 0, s \ne 0$, *then*

$$\left({}_0I_x^\alpha\right)\left[\,[r^2 cos_\alpha(x^\alpha) + s^2 sin_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right] = \frac{1}{rs} \cdot arctan_\alpha\left(\frac{s}{r} tan_\alpha(x^\alpha)\right), \quad (17)$$

and

$$\left({}_0I_x^\alpha\right)\left[\,[r^2 cos_\alpha(x^\alpha) - s^2 sin_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right] = \frac{1}{2rs} \cdot Ln_\alpha\left(\left|[s \cdot tan_\alpha(x^\alpha) + r] \otimes_\alpha [s \cdot tan_\alpha(x^\alpha) - r]^{\otimes_\alpha (-1)}\right|\right). \quad (18)$$

Proof $\quad \left({}_0I_x^\alpha\right)\left[\,[r^2 cos_\alpha(x^\alpha) + s^2 sin_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right]$

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$$= \left({}_0I_x^\alpha\right)\left[\left[r^2 + s^2 tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left[sec_\alpha(x^\alpha)\right]^{\otimes_\alpha 2}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[r^2 + s^2 tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[tan_\alpha(x^\alpha)\right]\right]$$

$$= \frac{1}{s^2}\left({}_0I_x^\alpha\right)\left[\left[\left(\frac{r}{s}\right)^2 + tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[tan_\alpha(x^\alpha)\right]\right]$$

$$= \frac{1}{s^2}\cdot\frac{s}{r}\cdot arctan_\alpha\left(\frac{s}{r} tan_\alpha\left(\frac{1}{2}x^\alpha\right)\right)$$

$$= \frac{1}{rs}\cdot arctan_\alpha\left(\frac{s}{r} tan_\alpha(x^\alpha)\right).$$

On the other hand,

$$\left({}_0I_x^\alpha\right)\left[\left[r^2 cos_\alpha(x^\alpha) - s^2 sin_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[r^2 - s^2 tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left[sec_\alpha(x^\alpha)\right]^{\otimes_\alpha 2}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\left[r^2 - s^2 tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[tan_\alpha(x^\alpha)\right]\right]$$

$$= \frac{1}{s^2}\left({}_0I_x^\alpha\right)\left[\left[\left(\frac{r}{s}\right)^2 - tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[tan_\alpha(x^\alpha)\right]\right]$$

$$= \frac{1}{s^2}\cdot\frac{s}{2r}\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{r}{s} + tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)} + \left[\frac{r}{s} - tan_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right] \otimes_\alpha \left({}_0D_x^\alpha\right)\left[tan_\alpha(x^\alpha)\right]\right]$$

$$= \frac{1}{2rs}\cdot\left[Ln_\alpha\left(\left|\frac{r}{s} + tan_\alpha(x^\alpha)\right|\right) - Ln_\alpha\left(\left|\frac{r}{s} - tan_\alpha(x^\alpha)\right|\right)\right]$$

$$= \frac{1}{2rs}\cdot Ln_\alpha\left(\left|\left[s\cdot tan_\alpha(x^\alpha) + r\right] \otimes_\alpha \left[s\cdot tan_\alpha(x^\alpha) - r\right]^{\otimes_\alpha(-1)}\right|\right).$$

Q.e.d.

Example 3.2: Let $0 < \alpha \le 1$, then

$$\left({}_0I_x^\alpha\right)\left[\left[9cos_\alpha(x^\alpha) + 4sin_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right] = \frac{1}{6}\cdot arctan_\alpha\left(\frac{2}{3} tan_\alpha(x^\alpha)\right). \quad (19)$$

And

$$\left({}_0I_x^\alpha\right)\left[\left[16cos_\alpha(x^\alpha) - 9sin_\alpha(x^\alpha)\right]^{\otimes_\alpha(-1)}\right] = \frac{1}{24}\cdot Ln_\alpha\left(\left|\left[3\cdot tan_\alpha(x^\alpha) + 4\right] \otimes_\alpha \left[3\cdot tan_\alpha(x^\alpha) - 4\right]^{\otimes_\alpha(-1)}\right|\right). \quad (20)$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we can obtain the solutions of some fractional integrals of fractional trigonometric functions. Moreover, our results are generalizations of the classical calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and engineering mathematics.

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