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Is there a relationship between the formula for the zeros of a polynomial and the coordinates of the centre of its graph?

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Abstract: **Let $p(x)$ be a polynomial of degree 1 or higher with real coefficients, and let R be the arithmetic mean (i.e., the average) of the zeros of $p(x)$. Then, we define the *centre* of the graph of p to be the point whose coordinates are $(R, p(R))$. In this paper, we provide a partial answer to the question posed in its title.**

1. INTRODUCTION

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (n \geq 1,\ a_n \neq 0)$ be an n^{th} degree polynomial with real coefficients, and let R be the arithmetic mean (i.e., the average) of the n zeros of p. Then, we define the *centre* of the graph of p to be the point whose coordinates are $(R,\ p(R))$. In [1], it is shown that $R = -\dfrac{a_{n-1}}{n a_n}$, and that for $n \geq 2$, the graph of p is more nearly symmetric at its centre than at any other point on the graph. Nickalls refers to the centre of the graph of a polynomial as the *N*-point in his papers [2], [3], and [4].

If p is a polynomial of degree 1, 2, or 3, the answer to the question posed in the title of this article is, unequivocally, "yes". In this case, the relationship can easily be made explicit. The authors conjecture that such a relationship holds for polynomials of degree 4 or higher, as well. Recall that the Abel–Ruffini theorem (also known as Abel's impossibility theorem) [5] states that there is no solution in radicals to general polynomial equations of degree 5 or higher with arbitrary coefficients. The theorem is named after Paolo Ruffini, who supplied an incomplete proof in 1799, and Niels Henrik Abel, who provided a proof in 1824.

2. POLYNOMIALS OF DEGREE 1

Suppose $p(x) = ax + b$, where $a \neq 0$. Then, the abscissa of the centre of the graph of p is $x = -\dfrac{b}{a}$.
Note that $p\left(-\dfrac{b}{a}\right) = a\left(-\dfrac{b}{a}\right) + b = 0$. Since p is linear, it has exactly one zero, namely, $x_1 = -\dfrac{b}{a}$.

3. POLYNOMIALS OF DEGREE 2

Let $p(x) = ax^2 + bx + c$, where $a \neq 0$. This time, the abscissa of the centre of the graph of p is $x = -\dfrac{b}{2a}$.

Observe that $p\left(-\dfrac{b}{2a}\right) = a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c$

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$$= \frac{-b^2 + 4ac}{4a}.$$

Since p is quadratic, it has exactly two zeros. According to the

Quadratic Formula, they are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. That is,

$$x_1 = -\frac{b}{2a} + \sqrt{\frac{(-1)^1}{a} \cdot p\left(-\frac{b}{2a}\right)} \quad \text{and} \quad x_2 = -\frac{b}{2a} - \sqrt{\frac{(-1)^1}{a} \cdot p\left(-\frac{b}{2a}\right)}.$$

4. POLYNOMIALS OF DEGREE 3

Suppose, now, that $p(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. In this case, the

abscissa of the centre of the graph of p is $x = -\dfrac{b}{3a}$. We have

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b}{3a}\right)^3 + b\left(-\frac{b}{3a}\right)^2 + c\left(-\frac{b}{3a}\right) + d$$

$$= \frac{2b^3 - 9abc + 27a^2 d}{27a^2}.$$

Also, since $p'(x) = 3ax^2 + 2bx + c$, $p'\left(-\dfrac{b}{3a}\right) = 3a\left(-\dfrac{b}{3a}\right)^2 + 2b\left(-\dfrac{b}{3a}\right) + c$

$$= \frac{-b^2 + 3ac}{3a}.$$

Since p is cubic, it has exactly three zeros. From the Cubic Formula [6], we know that one of these zeros (say, x_1)

can be expressed in the following form:

$$x_1 = -\frac{b}{3a}$$

$$+ \sqrt[3]{\frac{(-1)^1}{2a} \cdot p\left(-\frac{b}{3a}\right) + \sqrt{\left(\frac{(-1)^1}{2a} \cdot p\left(-\frac{b}{3a}\right)\right)^2 + \left(\frac{(-1)^2}{3a} \cdot p'\left(-\frac{b}{3a}\right)\right)^3}}$$

$$+ \sqrt[3]{\frac{(-1)^1}{2a} \cdot p\left(-\frac{b}{3a}\right) - \sqrt{\left(\frac{(-1)^1}{2a} \cdot p\left(-\frac{b}{3a}\right)\right)^2 + \left(\frac{(-1)^2}{3a} \cdot p'\left(-\frac{b}{3a}\right)\right)^3}}.$$

x_2 is obtained by multiplying the first cube root in the expression for x_1 by $e^{\frac{2\pi i}{3}}$ and the second cube root in the expression for x_1 by $e^{\frac{4\pi i}{3}}$. x_3 is obtained by multiplying the first cube root in the expression for x_1 by $e^{\frac{4\pi i}{3}}$ and the second cube root in the expression for x_1 by $e^{\frac{2\pi i}{3}}$.

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5. IS A GENERALIZABLE PATTERN BEGINNING TO EMERGE?

It appears that a complex, generalizable pattern does begin to emerge with respect to the above formulas for the zeros of linear, quadratic, and cubic functions. First of all, these formulas all involve the abscissa of the centre of the graph of p. Also, regarding the fraction that multiplies $p^{(r)}\left(-\dfrac{b}{na}\right)$ in the expressions for the zeros of quadratic and cubic functions, the exponent to which (-1) in the numerator is raised is always $r+1$, and the coefficient of a in the denominator is equal to the highest power of $p^{(r)}\left(-\dfrac{b}{na}\right)$ that appears in the expression.

6. POLYNOMIALS OF DEGREE 4 OR HIGHER

Unfortunately, the authors were unable to find a formula for the zeros of a quartic polynomial that involves fourth roots. The only formulas that we did find involve square roots [7], [8], so we were unable to verify that the pattern observed in the expressions for the zeros of linear, quadratic, and cubic polynomials persists for quartic polynomials. Nevertheless, as indicated in the introduction, the authors conjecture that there is a relationship between the formula for the zeros of a polynomial of degree 4 or higher and the coordinates of the centre of its graph. Evidence for the existence of such a relationship for quartic polynomials is provided in [9].

7. CONCLUSION

As the reader will immediately discover by visiting [7], [8], and [9], the formula for the zeros of a quartic polynomial is extremely unwieldy. The formula for the zeros of a polynomial of degree 5 or higher is downright monstrous. We, therefore, invite the interested reader in possession of substantial computer programming skills to either prove or disprove our conjecture.

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