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Relation between Canonical Sine Transform and Other Transforms

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Abstract: **As generalization of the fractional Cosine transform (FRCT), the canonical sine transform (CST) has been used in several areas, including optical analysis and signal processing. For practical purpose half canonical sine transform is more useful. Hence in this paper we have proved some important results about Inversion theorem for half canonical sine, Differentiation property, Modulation property, Parseval's Identity, Scaling property for half canonical sine transform.**

Keywords: **Linear canonical transform, Fractional Fourier Transform.**

1. INTRODUCTION

Last decades, since Namias in 1980 develop the eigenvalue methods for Fractional Fourier transform number of other integral transform have been extended in its fractional domain. For examples Almeida [1] had studied fractional Fourier transform, Fractional Hilbert transform has been developed by Zayed [7], Gudadhe, Joshi, [3] studied number of property of generalized half canonical sine transform etc. Bhosale and Choudhary [2] had studied it as a tempered distribution; number of applications of fractional Fourier transforms in signal processing, image processing filtering optics, etc is studied. These fractional transforms found number of applications in signal processing, image processing, quantum mechanics etc.

Further generalization of fractional Fourier transform known as linear canonical transform was introduced by Moshinsky [4] in 1971. Pei, Ding [5, 6] had studied its eigen value aspect. Linear canonical transform is a three parameter linear integral transform which has several special cases as fractional Fourier transform, Fresnel transform, Chirp transform etc. Linear canonical transform is defined as,

$$[LCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}(a/b)t^2} \cdot e^{-i(s/b)t} f(t)\, dt,$$

for $b \neq 0$

$$= \sqrt{d}\ e^{\frac{i}{2}(cd)s^2} \cdot f(d \cdot s),$$

for b = 0, with ad − bc =1,

Where a, b, c, and d are real parameters independent on s and t.

Pei and Ding [6] had defined canonical sine transform (CST) as

$$[CSTf(t)](s) = (-i)\sqrt{\frac{1}{2\pi ib}} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin\left(\frac{s}{b}t\right) f(t)\, dt$$

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1.1.1 Generalized Canonical Sine Transform (CST)]:

The Canonical Sine Transform $f \in \mathcal{E}'(R^n)$ can be defined by,

$$\{CST\ f(t)\}(s) = \langle f(t), K_S(t,s)\rangle$$

where, $K_S(t,s) = (-i)\sqrt{\frac{2}{\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right)$

Hence the generalized canonical sine transform of $f \in \mathcal{E}'(R^n)$ can be defined by,

$$\{CST\ f(t)\}(s) = \sqrt{\frac{1}{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} (-i)\sin\left(\frac{s}{b}t\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\, dt$$

1.1.2 If $\{CST\ f(t)\}(s)$ **denotes generalized Canonical Sine transform of** $f(t)$ **then**

$$\{CST\ f''(t)\}(s) = \left[i\frac{a}{b} + \frac{s^2}{b^2}\right]\{CST\ f(t)\}(s)$$

$$+i\frac{a^2}{b^2}\{CST\ t^2 f(t)\}(s) - \frac{2as}{b^2}\{CCT\ t\ f(t)\}(s)$$

Proof: We have,

$$\{CST\ f''(t)\} = (-i)\sqrt{\frac{1}{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\sin\left(\frac{s}{b}t\right) f''(t)\, dt$$

Integrating by parts,

$$\{CST\ f''(t)\} = (-i)\sqrt{\frac{1}{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \{[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\sin\left(\frac{s}{b}t\right) f'(t)]_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} [i\left(\frac{a}{b}\right)t\, e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\sin\left(\frac{s}{b}t\right) + \frac{s}{b} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right)]\ f'(t)dt\}$$

$$\{CST\ f''(t)\} = (-i)\left[\frac{a}{b}\{CST\ t\ f'(t)\}(s) + \frac{s}{b}\{CCT\ f'(t)\}(s)\right]$$

$$= (-i)\{-i\sqrt{\frac{1}{2\pi ib}}\left(\frac{a}{b}\right)\{\left[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right) t\ f(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [\{e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right)$$

$$+i\frac{a}{b}t^2 e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right) + \frac{s}{b} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} t\cos\left(\frac{s}{b}t\right)]\ f(t)dt\}$$

$$-i\frac{s}{b}\sqrt{\frac{1}{2\pi ib}}\{\left[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right) f(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [i\frac{a}{b}te^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right)$$

$$-\frac{s}{b} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right)]f(t)dt\}$$

$$\{CST\ f''(t)\}(s) = i\frac{a}{b}\{CST\ f(t)\}(s) + i\frac{a^2}{b^2}\{CST\ t^2 f(t)\}(s)$$

$$-\frac{as}{b^2}\{CCT\, t\ f(t)\}(s) - \frac{as}{b^2}\{CCT\ t\ f(t)\}(s) + \frac{s^2}{b^2}\{CST\ f(t)\}(s)$$

$$\{CST\ f''(t)\}(s) = \left[i\frac{a}{b} + \frac{s^2}{b^2}\right]\{CST\ f(t)\}(s)$$

$$+i\frac{a^2}{b^2}\{CST\ t^2 f(t)\}(s) - \frac{2as}{b^2}\{CCT\ t\ f(t)\}(s)$$

1.1.3 If $\{CST\ f(t)\}(s)$ **denotes generalized Canonical Sine transform of** $f(t)$ **then**

$$\{CST\ e^{izt}\ f(t)\}(s) = \{CST \cos zt\ f(t)\}(s) + i\{CST \sin zt\ f(t)\}(s)$$

Proof: We have, $\{CST\ e^{izt}\ f(t)\}(s) = -i\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sin\left(\frac{s}{b}t\right)e^{izt}\ f(t)\,dt$

$$= -i\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sin\left(\frac{s}{b}t\right)(\cos zt + i\sin zt)\ f(t)\,dt$$

$$= -i\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(\sin\left(\frac{s}{b}t\right)\cos zt + i\sin\left(\frac{s}{b}t\right)\sin zt\right) f(t)\,dt$$

$$\left\{CST\ e^{izt}\ f(t)\right\}(s) = \{CST\cos zt\ f(t)\}(s) + i\{CST\sin zt\ f(t)\}(s)$$

1.1.4 Shifting in s-Plane:

If $\left\{CST\ \ f(t)\right\}(s)$ denotes generalized Canonical Sine transform of $f(t)$ and 'ω, is any real number. then,

$$\{CST[f(t)]\}(s+\omega) = e^{\frac{i}{2}\left(\frac{d}{b}\right)(\omega^2+2s\omega)}\left[\left\{CST[\cos\left(\frac{\omega}{b}t\right)f(t)]\right\}(s) - i\left\{CCT[\sin\left(\frac{\omega}{b}t\right)f(t)]\right\}(s)\right]$$

Proof: We have,

$$\{CST\ \ f(t)\}(s+\omega) = -i\sqrt{\frac{1}{2\pi ib}}\,e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+\omega)^2}\int_{-\infty}^{\infty}\sin\left(\frac{(s+\omega)}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ f(t)\,dt$$

$$= -i\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)(\omega^2+2s\omega)}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\left[\sin\left(\frac{s}{b}t\right)\cos\left(\frac{\omega}{b}t\right) + \cos\left(\frac{s}{b}t\right)\sin\left(\frac{\omega}{b}t\right)\right]e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ f(t)\,dt$$

$$= e^{\frac{i}{2}\left(\frac{d}{b}\right)(\omega^2+2s\omega)}\Big\{-i\sqrt{\frac{1}{2\pi ib}}\,e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\sin\left(\frac{s}{b}t\right)\cos\left(\frac{\omega}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ f(t)\,dt - i\sqrt{\frac{1}{2\pi ib}}\,e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)\sin\left(\frac{\omega}{b}t\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ f(t)\,dt\Big\}$$

$$\{CST[f(t)]\}(s+\omega) = e^{\frac{i}{2}\left(\frac{d}{b}\right)(\omega^2+2s\omega)}\left[\left\{CST[\cos\left(\frac{\omega}{b}t\right)f(t)]\right\}(s) - i\left\{CCT[\sin\left(\frac{\omega}{b}t\right)f(t)]\right\}(s)\right]$$

1.1.5 Relation between Canonical Sine Transform and Fractional Sine Transform:

If $A = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then CST change into Fractional Fourier Sine transform.

Proof: By definition of CST,

$$\{CST\ f(t)\}(s) = (-i)\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} sin\left(\frac{s}{b}t\right)f(t)dt$$

For $A = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ we get,

$$= (-i)\sqrt{\frac{1}{2\pi i\sin\alpha}}\cdot e^{\frac{i}{2}(\cot\alpha)s^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}(\cot\alpha)t^2}\sin(\operatorname{cosec}\alpha\ st)\,f(t)\,dt$$

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Multiplying both sides by $e^{\frac{1}{2}i\alpha}$,

We get,

$$e^{\frac{1}{2}i\alpha}\{CST\ f(t)\}(s)=(-i)\sqrt{\cos\alpha+i\sin\alpha}\sqrt{\frac{1}{2\pi i\sin\alpha}}\int_{-\infty}^{\infty}e^{\frac{i}{2}(s^2+t^2)\cot\alpha}\sin(\operatorname{cosec}\alpha\ st)\,f(t)\,dt=(-i)\sqrt{\frac{\cos\alpha+i\sin\alpha}{2\pi i\sin\alpha}}\int_{-\infty}^{\infty}e^{\frac{i}{2}(s^2+t^2).\cot\alpha}\sin(\operatorname{cosec}\alpha\ st)\,f(t)\,dt$$

$$=(-i)\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}e^{\frac{i}{2}(s^2+t^2).\cot\alpha}\sin(\operatorname{cosec}\alpha\ st)\,f(t)\,dt \qquad e^{\frac{1}{2}i\alpha}\{CSTf(t)\}(s)=[FrFSTf(t)](s)$$

This is the Fractional Fourier Sine transform.

1.1.6 Relation Between Canonical Sine Transform (CST) and Simplified Fractional Sine Transform:

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}\cot\phi & 1\\ -1 & 0\end{bmatrix},(\phi=\frac{\alpha\pi}{2})$ then CST change into Simplified Fractional Fourier Sine transform (SFRFST)

Proof: By definition of CST $\{CST\ f(t)\}(s)=(-i)\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sin\left(\frac{s}{b}t\right)f(t)dt$

For $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}\cot\phi & 1\\ -1 & 0\end{bmatrix}$, $(\phi=\frac{\alpha\pi}{2})$ we get, $\{CST\ f(t)\}(s)=\left((-i)\sqrt{\frac{1}{2\pi i}}\int_{-\infty}^{\infty}e^{\frac{i}{2}(\cot\phi).t^2}\sin(st)\,f(t)\,dt\right)$

We suppose that the input $f(t)$ is a real function $\{CST\ f(t)\}(s)=\left(\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\sin(\frac{1}{2}\cot\phi\ t^2)\sin(st)\,f(t)\,dt\right)$

$=\{F_{SS}^{\alpha}[f(t)]\}(s)$

This is the Simplified Fractional Fourier Sine transform

2. CONCLUSION

In this paper, brief introduction of the generalized canonical sine transform is given and its relationship between canonical sine transform and other transforms which will be useful in solving differential equations occurring in signal processing and many other branches of engineering.

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