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Study of a Problem Involving Fractional Exponential Equation

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***Abstract*: In this paper, based on a new multiplication of fractional analytic functions, we study a problem involving fractional exponential equation. We can find the solution of this problem by using some techniques. In fact, this problem is a generalization of the traditional exponential equation problem.**

***Keywords*: New multiplication, Fractional analytic functions, Fractional exponential equation.**

I. INTRODUCTION

Fractional calculus includes the derivative and integral of any real order or complex order. In the past few decades, fractional calculus has gained much attention as a result of its demonstrated applications in various fields of science and engineering such as physics, biology, mechanics, electrical engineering, viscoelasticity, dynamics, control theory, modelling, economics, and so on [1-11].

In this paper, based on a new multiplication of fractional analytic functions, a problem involving fractional exponential equation is studied. Using some methods, the solution of this problem can be obtained. On the other hand, our result is a generalization of the ordinary exponential equation result.

II. PRELIMINARIES

Definition 2.1 ([12])**:** If x and a_k are real numbers for all k, and $0 < \alpha \leq 1$. Suppose that the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic function.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.2 ([13])**:** Suppose that $0 < \alpha \leq 1$. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}, \quad (1)$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \quad (2)$$

Then

$$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right) x^{k\alpha}. \quad (3)$$

Equivalently,

$$f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)$$

$$=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \quad (4)$$

Definition 2.3 ([14])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes n}=f_\alpha(x^\alpha)\otimes\cdots\otimes f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.

Definition 2.4 ([15])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}, \quad (5)$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \quad (6)$$

We define the compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \quad (7)$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\left(f_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \quad (8)$$

Definition 2.5 ([16])**:** Let $0<\alpha\leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (9)$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.6 ([16])**:** If $0<\alpha\leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \quad (10)$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Definition 2.7 ([17])**:** Let $0<\alpha\leq 1$. If $u_\alpha(x^\alpha)$, $w_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then the α-fractional power exponential function $u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)}$ is defined by

$$u_\alpha(x^\alpha)^{\otimes w_\alpha(x^\alpha)}=E_\alpha\left(w_\alpha(x^\alpha)\otimes Ln_\alpha\left(u_\alpha(x^\alpha)\right)\right). \quad (11)$$

Definition 2.8 ([18])**:** Let $0<\alpha\leq 1$, and $a_\alpha>0, a_\alpha\neq 1$. Then

$$a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha}=E_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes Ln_\alpha(a_\alpha)\right)=E_\alpha\left(Ln_\alpha(a_\alpha)\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \quad (12)$$

is called the α-fractional exponential function based on a_α .

Definition 2.9 ([18])**:** Let $0<\alpha\leq 1$, and $a_\alpha>0, a_\alpha\neq 1$. Then we define $Log_{a_\alpha}(x^\alpha)$ is the inverse function of $a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha}$. In particular, $Log_{e_\alpha}(x^\alpha)=Ln_\alpha(x^\alpha)$.

Proposition 2.10 ([18])**:** *If* $0<\alpha\leq 1$, *and* $a_\alpha>0$, $a_\alpha\neq 1$. *Then*

$$a_\alpha^{\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)}=a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha}\otimes a_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}y^\alpha}. \quad (13)$$

$$Log_{a_\alpha}(x^\alpha\otimes y^\alpha)=Log_{a_\alpha}(x^\alpha)+Log_{a_\alpha}(y^\alpha). \quad (14)$$

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III. MAIN RESULT

In this section, we solve a problem involving fractional exponential equation. At first, a lemma is needed.

Lemma 3.1: *If* $0 < \alpha \leq 1$, $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions, $f_\alpha(x^\alpha) > 0$, $g_\alpha(x^\alpha) > 0$, *and* $f_\alpha(x^\alpha), g_\alpha(x^\alpha) \neq 1$, *then*

$$\left[Log_{g_\alpha(x^\alpha)}\left(f_\alpha(x^\alpha)\right)\right]^{\otimes -1} = Log_{f_\alpha(x^\alpha)}\left(g_\alpha(x^\alpha)\right). \tag{15}$$

Proof Let $Log_{g_\alpha(x^\alpha)}\left(f_\alpha(x^\alpha)\right) = \varphi_\alpha(x^\alpha)$, then

$$g_\alpha(x^\alpha)^{\otimes \varphi_\alpha(x^\alpha)} = f_\alpha(x^\alpha). \tag{16}$$

Let $Log_{f_\alpha(x^\alpha)}\left(g_\alpha(x^\alpha)\right) = \rho_\alpha(x^\alpha)$, then

$$f_\alpha(x^\alpha)^{\otimes \rho_\alpha(x^\alpha)} = g_\alpha(x^\alpha). \tag{17}$$

Therefore,

$$\left[\, g_\alpha(x^\alpha)^{\otimes \varphi_\alpha(x^\alpha)}\right]^{\otimes \rho_\alpha(x^\alpha)} = g_\alpha(x^\alpha). \tag{18}$$

Hence,

$$g_\alpha(x^\alpha)^{\otimes[\varphi_\alpha(x^\alpha)\otimes\rho_\alpha(x^\alpha)]} = g_\alpha(x^\alpha)$$

Thus,

$$\varphi_\alpha(x^\alpha)\otimes\rho_\alpha(x^\alpha) = 1. \tag{19}$$

That is,

$$[\varphi_\alpha(x^\alpha)]^{\otimes -1} = \rho_\alpha(x^\alpha).$$

Finally, we get

$$\left[Log_{g_\alpha(x^\alpha)}\left(f_\alpha(x^\alpha)\right)\right]^{\otimes -1} = Log_{f_\alpha(x^\alpha)}\left(g_\alpha(x^\alpha)\right).$$

Q.e.d.

Problem 3.2: Let $0 < \alpha \leq 1$, $p_\alpha, q_\alpha, r_\alpha, s, t, x, y$ be real numbers, $p_\alpha > 0, q_\alpha > 0, r_\alpha > 0$, and $p_\alpha, q_\alpha, r_\alpha \neq 1$.If the α-fractional exponential equation holds:

$$p_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha} = q_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}y^\alpha} = r_\alpha. \tag{20}$$

Find

$$s\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes -1} + t\cdot\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes -1}. \tag{21}$$

Solution Since $p_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha} = r_\alpha$, it follows that

$$Log_{\,p_\alpha}\left(p_\alpha^{\otimes\frac{1}{\Gamma(\alpha+1)}x^\alpha}\right) = Log_{\,p_\alpha}(r_\alpha). \tag{22}$$

That is,

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha = Log_{\,p_\alpha}(r_\alpha). \tag{23}$$

By Lemma 3.1,

$$\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes -1} = Log_{r_\alpha}(p_\alpha). \tag{24}$$

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Similarly, since $q_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} y^\alpha} = r_\alpha$, we have

$$Log_{q_\alpha}\left(q_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} y^\alpha} \right) = Log_{q_\alpha}(r_\alpha). \quad (25)$$

Thus,

$$\frac{1}{\Gamma(\alpha+1)} y^\alpha = Log_{q_\alpha}(r_\alpha). \quad (26)$$

Also by Lemma 3.1, we obtain

$$\left[\frac{1}{\Gamma(\alpha+1)} y^\alpha\right]^{\otimes -1} = Log_{r_\alpha}(q_\alpha). \quad (27)$$

Finally, we get

$$\begin{aligned} & s \cdot \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]^{\otimes -1} + t \cdot \left[\frac{1}{\Gamma(\alpha+1)} y^\alpha\right]^{\otimes -1} \\ & = s \cdot Log_{r_\alpha}(p_\alpha) + t \cdot Log_{r_\alpha}(q_\alpha) \\ & = Log_{r_\alpha}({p_\alpha}^s) + Log_{r_\alpha}({q_\alpha}^t) \\ & = Log_{r_\alpha}({p_\alpha}^s \cdot {q_\alpha}^t) \quad . \end{aligned} \quad (28)$$

IV. CONCLUSION

Based on a new multiplication of fractional analytic functions, this paper studies a problem involving fractional exponential equation. In fact, the fractional exponential equation is a generalization of ordinary exponential equation. Using some techniques, we can obtain the solution of this problem. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and applied mathematics.

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