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The Harmonic Green Function for a Right Isosceles Triangle

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Abstract: **in this paper, we have constructed the harmonic green function for a right isosceles triangle in the complex plane, by using the reflections over it segments providing parqueting to the complex plane.**

Keywords: **complex plane, elliptic functions, green function, parquetting, reflection, right triangle.**

1. HARMONIC GREEN FUNCTION FOR A REGULAR DOMAIN

The harmonic green function for a regular domain D (bounded domain) is the fundamental solution of the inhomogeneous Laplace's equation $\Delta u = f$ where

$f \in L_p(D,\mathbb{C}), 2 < p$ with vanishing values on the boundary, there are three different methods to find the harmonic green function,

the first one by using conformal invariance $w: D \to \Omega$ [1] that maps the domain D to another domain Ω that we already know the harmonic green function in it $G_{1\Omega}$, hence

$$G_{1D}(z,\zeta) = G_{1\Omega}\big(w(z), w(\zeta)\big).$$

The second method by solving the Schwarz problem for analytic functions, [1] where the Schwarz kernel of the domain D must be found to solve the Schwarz problem and obtain the harmonic green function of the domain D.

The last method by using reflections along the boundary of the regular domain D, this method is effective to get the harmonic green functions for some domains that can provide a parqueting for the complex plane or circular arcs as half circle, ring and half ring [2], etc.

The principal of this method starting with a fixed point $\zeta \in D$ and a vary point

$z \in D\backslash\{\zeta\}$,we begin to reflect z along the borders of D having an elliptic function $B(z,\zeta)$ with the double periods

$$\Omega_{m,n} = mw_1 + nw_1.$$

Where $w_1, w_2 \in \mathbb{C}$ and $\frac{w_1}{w_2} \notin \mathbb{R}$, $m, n \in \mathbb{Z}$

It turns out that the harmonic green function for D is $\log|B(z,\zeta)|^2$.

See [1] for a strip $S = \{z \in \mathbb{C}; 0 \le Imz \le \pi\}$ and a rectangle, [3] for equilateral triangle, and [4] for the quarter ring and half hexagon.

Notice that if the elliptic function represented as an infinite product we have to proof the convergence.

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2. THE RIGHT ISOSCELES TRIANGLE

To construct the harmonic green function $G_1(z,\zeta)$ for the triangle T with the corners $0 , 1 , i$, we start to reflect T over its three segments and continuing to reflect the resulting triangles over its segments having a parqueting to the complex plane.

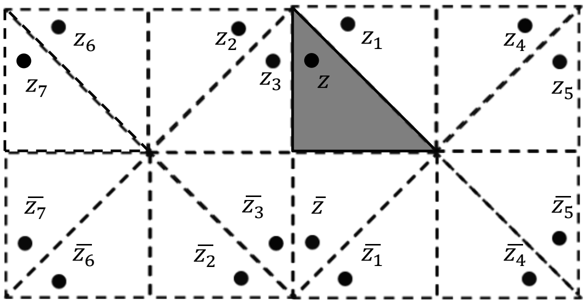

Figure 1

Let $\in T$, reflecting z at the segment from 1 to i gives

$$z_1 = -i\bar{z} + 1 + i \,.$$

So the reflection of T at the segment from 1 to i gives the triangle T_1

with the corners $1 , 1 + i , i$.

Reflecting z at the segment from i to 0 gives

$$z_3 = -\bar{z} \,.$$

So the reflection of T at the segment from i to 0 gives the triangle T_3

with the corners $0 , i , -1$.

Reflecting z_3 at the segment from i to -1 gives

$$z_2 = -iz - 1 + i \,.$$

So the reflection of T_3 at the segment from -1 to i gives the triangle T_2

With the corners $-1 - i , i , -1$.

By continuing we obtain the points

$$z_4 = -iz + 1 + i \,,$$

$$z_5 = -\bar{z} + 2 \,,$$

$$z_6 = -i\bar{z} - 1 + i \,,$$

$$z_7 = z - 2 \,.$$

Reflecting the eight points at the real axis we get the points

$$\bar{z} , \bar{z}_1 , \bar{z}_2 , \bar{z}_3 , \bar{z}_4 , \bar{z}_5 , \bar{z}_6 , \bar{z}_7 \,.$$

Remark: all of the obtained points is a result from the point , after applying suitable rotation and shifting.

Denoting $\Omega_{m,n} = 2m + 2ni \;, m, n \in \mathbb{Z}$.

We can express any point of the complex plane $\check{z}$ by using one of the points

$$z , z_1 , z_2 , z_3 , \bar{z} , \bar{z}_1 , \bar{z}_2 , \bar{z}_3 \,.$$

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With a proper shifting Ω_{m_0,n_0} , that gives

$\check{z} = z_k + \Omega_{m_0,n_0}$ or $\check{z} = \overline{z_k} + \Omega_{m_0,n_0}$; $0 \le k \le 3$, $z = z_0$.

So we get the following elliptic function

$$B(z,\zeta) = \prod_{m,n\in\mathbb{Z}} \frac{\zeta - \bar{z} - \Omega_{m,n}}{\zeta - z - \Omega_{m,n}} \cdot \frac{\zeta - z_1 - \Omega_{m,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \cdot \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - z_2 - \Omega_{m,n}} \cdot \frac{\zeta - z_3 - \Omega_{m,n}}{\zeta - \overline{z_3} - \Omega_{m,n}}$$

where z is a variable point in D and $\zeta \in D$ is fixed.

Remark: we can see that z and every direct reflection of $\bar{z}$ to all directions makes a simple poles for the function $B(z,\zeta)$.

Similarly, $\bar{z}$ and every direct reflection of z to all directions makes zeros for the function $B(z,\zeta)$,see [5],hence

$$G_1(z,\zeta) = \log|B(z,\zeta)\,|^2 = log \prod_{m,n\in\mathbb{Z}} \left| \frac{\zeta - \bar{z} - \Omega_{m,n}}{\zeta - z - \Omega_{m,n}} \cdot \frac{\zeta - z_1 - \Omega_{m,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \cdot \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - z_2 - \Omega_{m,n}} \cdot \frac{\zeta - z_3 - \Omega_{m,n}}{\zeta - \overline{z_3} - \Omega_{m,n}} \right|^2 .$$

Theorem (1):

The function $G_1(z,\zeta)$ is the Green function for the right isosceles triangle T satisfying:

- $G_1(.,\zeta)$ is harmonic in $T\backslash\{\zeta\}$,
- $G_1(.,\zeta) + log|\zeta - z|^2$ is harmonic in T ,
- $\lim_{z\to\partial T} G_1(z,\zeta) = 0$,
- $G_1(z,\zeta) = G_1(\zeta,z)$ for $z,\zeta \in T$,

for any $\zeta \in T$.

The proof of theorem 1 holds in the three following lemmas.

Lemma (1): The double infinite products

$$\prod_{m,n\in\mathbb{Z}} \left| \frac{\zeta - \overline{z_k} - \Omega_{m,n}}{\zeta - z_k - \Omega_{m,n}} \right|^2$$

converge for $0 \le k \le 3$, where $z_0 = z \in T$.

Proof:

Let $a_{m,n} = \left| \frac{\zeta - \overline{z_k} - \Omega_{m,n}}{\zeta - z_k - \Omega_{m,n}} \right|^2$ then rewriting the double infinite product gives

$$\prod_{m,n\in\mathbb{Z}} a_{m,n} = a_{0,0} \prod_{m=1}^{\infty} a_{m,0}.\, a_{-m,0} \prod_{n=1}^{\infty} a_{0,n}.\, a_{0,-n} \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} a_{m,n}.\, a_{-m,n}.\, a_{m,-n}.\, a_{-m,-n}$$

We have

$$\prod_{m=1}^{\infty} a_{m,0}.\, a_{-m,0} = \prod_{m=1}^{\infty} \left| \frac{\zeta - \overline{z_k} - 2m}{\zeta - z_k - 2m} \cdot \frac{\zeta - \overline{z_k} + 2m}{\zeta - z_k + 2m} \right|^2 = \prod_{m=1}^{\infty} \left| \frac{(\zeta - \overline{z_k})^2 - 4m^2}{(\zeta - z_k)^2 - 4m^2} \right|^2$$

The convergence of the last product as the convergence of the series:

$$\sum_{m=1}^{\infty} \left[\frac{4m^2 - (\zeta - \overline{z_k})^2}{4m^2 - (\zeta - z_k)^2} - 1 \right] = \sum_{m=1}^{\infty} \left[\frac{(\zeta - z_k)^2 - (\zeta - \overline{z_k})^2}{4m^2 - (\zeta - z_k)^2} \right]$$

And it’s convergent.

Similarly

$$\prod_{m=1}^{\infty} a_{0,n}.a_{0,-n} = \prod_{m=1}^{\infty}\left|\frac{\zeta-\overline{z_k}-2ni}{\zeta-z_k-2ni}.\frac{\zeta-\overline{z_k}+2ni}{\zeta-z_k+2ni}\right|^2 = \prod_{m=1}^{\infty}\left|\frac{(\zeta-\overline{z_k})^2+4n^2}{(\zeta-z_k)^2+4n^2}\right|^2$$

The convergence of the last product as the convergence of the series:

$$\sum_{m=1}^{\infty}\left[\frac{4n^2+(\zeta-\overline{z_k})^2}{4n^2+(\zeta-z_k)^2}-1\right] = \sum_{m=1}^{\infty}\left[\frac{(\zeta-\overline{z_k})^2-(\zeta-z_k)^2}{4n^2+(\zeta-z_k)^2}\right]$$

And it's convergent.

On the other hand, we have

$$\prod_{m=1}^{\infty}\prod_{n=1}^{\infty} a_{m,n}.a_{-m,n}.a_{m,-n}.a_{-m,-n} = \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{\zeta-\overline{z_k}-\Omega_{m,n}}{\zeta-z_k-\Omega_{m,n}}.\frac{\zeta-\overline{z_k}-\Omega_{-m,n}}{\zeta-z_k-\Omega_{-m,n}}\frac{\zeta-\overline{z_k}-\Omega_{m,-n}}{\zeta-z_k-\Omega_{m,-n}}\frac{\zeta-\overline{z_k}-\Omega_{-m,-n}}{\zeta-z_k-\Omega_{-m,-n}}\right|^2$$

$$= \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{\zeta-\overline{z_k}-\Omega_{m,n}}{\zeta-z_k-\Omega_{m,n}}.\frac{\zeta-\overline{z_k}+\overline{\Omega_{m,n}}}{\zeta-z_k+\overline{\Omega_{m,n}}}.\frac{\zeta-\overline{z_k}-\overline{\Omega_{m,n}}}{\zeta-z_k-\overline{\Omega_{m,n}}}.\frac{\zeta-\overline{z_k}+\Omega_{m,n}}{\zeta-z_k+\Omega_{m,n}}\right|^2$$

$$= \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{(\zeta-\overline{z_k})^2-\Omega^2{}_{m,n}}{(\zeta-z_k)^2-\Omega^2{}_{m,n}}.\frac{(\zeta-\overline{z_k})^2-\overline{\Omega_{m,n}}^2}{(\zeta-z_k)^2-\overline{\Omega_{m,n}}^2}\right|^2$$

$$= \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{(\zeta-\overline{z_k})^4-2(4m^2-4n^2)(\zeta-\overline{z_k})^2+(4m^2+4n^2)^2}{(\zeta-z_k)^4-2(4m^2-4n^2)(\zeta-z_k)^2+(4m^2+4n^2)^2}\right|^2$$

converge as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{(\zeta-\overline{z_k})^4-2(4m^2-4n^2)(\zeta-\overline{z_k})^2+(4m^2+4n^2)^2}{(\zeta-z_k)^4-2(4m^2-4n^2)(\zeta-z_k)^2+(4m^2+4n^2)^2}-1\right]$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{16(m^2+n^2)^2-8(m^2-n^2)(\zeta-\overline{z_k})^2+(\zeta-\overline{z_k})^4}{16(m^2+n^2)^2-8(m^2-n^2)(\zeta-z_k)^2+(\zeta-z_k)^4}-1\right]$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{8(m^2-n^2)[(\zeta-z_k)^2-(\zeta-\overline{z_k})^2]+(\zeta-\overline{z_k})^4-(\zeta-z_k)^4}{16(m^2+n^2)^2-8(m^2-n^2)(\zeta-z_k)^2+(\zeta-z_k)^4}\right]$$

And it convergence.

Lemma (2): The function $G_1(.,\zeta)$ has vanishing boundary values on ∂T , for $\zeta \in T$.

$$\lim_{\substack{z\to z_0\in\partial T\\ z\in T}} G_1(z,\zeta) = 0$$

Proof:

we have to investigate all of the three segments $\partial_1 T, \partial_2 T, \partial_3 T$:

i. on the segment $\partial_1 T$ from 0 to 1 ,where :

$z = \bar{z}$, $z_2 = z_1 - 2$, $z_3 = -z$.

That gives

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$$\frac{\zeta-\bar{z}-\Omega_{m,n}}{\zeta-z-\Omega_{m,n}}\cdot\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\cdot\frac{\zeta-\overline{z_2}-\Omega_{m,n}}{\zeta-z_2-\Omega_{m,n}}\cdot\frac{\zeta-z_3-\Omega_{m,n}}{\zeta-\overline{z_3}-\Omega_{m,n}}$$
$$=\frac{\zeta-z-\Omega_{m,n}}{\zeta-z-\Omega_{m,n}}\cdot\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\cdot\frac{\zeta-\overline{z_1}+2-\Omega_{m,n}}{\zeta-z_1+2-\Omega_{m,n}}\cdot\frac{\zeta+z-\Omega_{m,n}}{\zeta+z-\Omega_{m,n}}$$
$$=\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\cdot\frac{\zeta-\overline{z_1}+2-\Omega_{m,n}}{\zeta-z_1+2-\Omega_{m,n}}$$
$$=\frac{\zeta-\overline{z_1}-\Omega_{m-1,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\cdot\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-z_1-\Omega_{m-1,n}}$$

We have the double product

$$\prod_{m,n\in\mathbb{Z}}\left|\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-z_1-\Omega_{m-1,n}}\right|^2$$

converges , so we can write

$$\prod_{m\in\mathbb{Z}}\left|\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-z_1-\Omega_{m-1,n}}\right|^2=\lim_{M\to\infty}\prod_{m=-M}^{+M}\left|\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-z_1-\Omega_{m-1,n}}\right|^2$$
$$=\lim_{M\to\infty}\left|\frac{\zeta-z_1-\Omega_{M,n}}{\zeta-z_1-\Omega_{-M-1,n}}\right|^2=\lim_{M\to\infty}\left|\frac{\zeta-z_1-2M-2ni}{\zeta-z_1+2M+2-2ni}\right|^2=1$$

On the other hand, and because the double product

$$\prod_{m,n\in\mathbb{Z}}\left|\frac{\zeta-\overline{z_1}-\Omega_{m-1,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\right|^2$$

is convergent, we can write

$$\prod_{m\in\mathbb{Z}}\left|\frac{\zeta-\overline{z_1}-\Omega_{m-1,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\right|^2=\lim_{M\to\infty}\prod_{m=-M}^{+M}\left|\frac{\zeta-\overline{z_1}-\Omega_{m-1,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\right|^2$$
$$=\lim_{M\to\infty}\left|\frac{\zeta-\overline{z_1}-\Omega_{M-1,n}}{\zeta-\overline{z_1}-\Omega_{-M,n}}\right|^2=\lim_{M\to\infty}\left|\frac{\zeta-\overline{z_1}-2M+2-2ni}{\zeta-\overline{z_1}+2M-2ni}\right|^2=1$$

That gives

$$\lim_{\substack{z\to z_0\in\partial_1 T\\ z\in T}} G_1(z,\zeta)=0\ .$$

ii. on the segment $\partial_2 T$ from 1 to i where :

$$z=z_1\ \ ,\ z_2=z_3\ .$$

That gives

$$\frac{\zeta-\bar{z}-\Omega_{m,n}}{\zeta-z-\Omega_{m,n}}\cdot\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}}\cdot\frac{\zeta-\overline{z_2}-\Omega_{m,n}}{\zeta-z_2-\Omega_{m,n}}\cdot\frac{\zeta-z_3-\Omega_{m,n}}{\zeta-\overline{z_3}-\Omega_{m,n}}$$
$$=\frac{\zeta-\bar{z}-\Omega_{m,n}}{\zeta-z-\Omega_{m,n}}\cdot\frac{\zeta-z-\Omega_{m,n}}{\zeta-\bar{z}-\Omega_{m,n}}\cdot\frac{\zeta-\overline{z_2}-\Omega_{m,n}}{\zeta-z_2-\Omega_{m,n}}\cdot\frac{\zeta+z_2-\Omega_{m,n}}{\zeta+\overline{z_2}-\Omega_{m,n}}=1$$

Thus

$$\lim_{\substack{z\to z_0\in\partial_2 T\\ z\in T}} G_1(z,\zeta)=0\ .$$

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iii. on the segment $\partial_3 T$ from i to 0 where :

$$z_3 = z \ , \ z_2 = \overline{z_2} + 2 \ , \ z_1 = \overline{z_1} + 2 \, .$$

We have

$$\frac{\zeta - \bar{z} - \Omega_{m,n}}{\zeta - z - \Omega_{m,n}} \cdot \frac{\zeta - z_3 - \Omega_{m,n}}{\zeta - \overline{z_3} - \Omega_{m,n}} = \frac{\zeta - \bar{z} - \Omega_{m,n}}{\zeta - z - \Omega_{m,n}} \cdot \frac{\zeta - z - \Omega_{m,n}}{\zeta - \bar{z} - \Omega_{m,n}} = 1$$

$$\frac{\zeta - z_1 - \Omega_{m,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \cdot \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - z_2 - \Omega_{m,n}} = \frac{\zeta - \overline{z_1} - 2 - \Omega_{m,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \cdot \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - \overline{z_2} - 2 - \Omega_{m,n}}$$

$$= \frac{\zeta - \overline{z_1} - \Omega_{m+1,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \cdot \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - \overline{z_2} - \Omega_{m+1,n}}$$

The double products

$$\prod_{m,n \in \mathbb{Z}} \left| \frac{\zeta - \overline{z_1} - \Omega_{m+1,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \right|^2$$

converges, so

$$\prod_{m \in \mathbb{Z}} \left| \frac{\zeta - \overline{z_1} - \Omega_{m+1,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \right|^2 = \lim_{M \to \infty} \prod_{m=-M}^{+M} \left| \frac{\zeta - \overline{z_1} - \Omega_{m+1,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \right|^2$$

$$= \lim_{M \to \infty} \left| \frac{\zeta - \overline{z_1} - \Omega_{M+1,n}}{\zeta - \overline{z_1} - \Omega_{M,n}} \right|^2 = \lim_{M \to \infty} \left| \frac{\zeta - \overline{z_1} - 2M - 2 - 2ni}{\zeta - \overline{z_1} - 2M - 2ni} \right|^2 = 1$$

By using the same technic on the double products

$$\prod_{m,n \in \mathbb{Z}} \left| \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - \overline{z_2} - \Omega_{m+1,n}} \right|^2$$

we obtain

$$\lim_{\substack{z \to z_0 \in \partial_3 T \\ z \in T}} G_1(z, \zeta) = 0 \, .$$

<u>Lemma (3):</u> For $z, \zeta \in T$ the symmetry relation

$$G_1(z, \zeta) = G_1(\zeta \, , z)$$

Holds.

Proof:

$$G_1(z, \zeta) = \prod_{m,n \in \mathbb{Z}} \left| \frac{\zeta - \bar{z} - \Omega_{m,n}}{\zeta - z - \Omega_{m,n}} \cdot \frac{\zeta - z_1 - \Omega_{m,n}}{\zeta - \overline{z_1} - \Omega_{m,n}} \cdot \frac{\zeta - \overline{z_2} - \Omega_{m,n}}{\zeta - z_2 - \Omega_{m,n}} \cdot \frac{\zeta - z_3 - \Omega_{m,n}}{\zeta - \overline{z_3} - \Omega_{m,n}} \right|^2$$

$$= \prod_{m,n \in \mathbb{Z}} \left| \frac{z - \bar{\zeta} + \overline{\Omega_{m,n}}}{z - \zeta + \Omega_{m,n}} \cdot \frac{z_1 - \zeta + \Omega_{m,n}}{\overline{z_1} - \zeta + \Omega_{m,n}} \cdot \frac{z_2 - \bar{\zeta} + \overline{\Omega_{m,n}}}{z_2 - \zeta + \Omega_{m,n}} \cdot \frac{z_3 - \zeta + \Omega_{m,n}}{\overline{z_3} - \zeta + \Omega_{m,n}} \right|^2$$

First, we have

$$\prod_{m,n \in \mathbb{Z}} \left| \frac{z_1 - \zeta + \Omega_{m,n}}{\overline{z_1} - \zeta + \Omega_{m,n}} \right|^2 = \prod_{m,n \in \mathbb{Z}} \left| \frac{1 + i - i\bar{z} - \zeta + \Omega_{m,n}}{1 - i + iz - \zeta + \Omega_{m,n}} \right|^2$$

$$= \prod_{m,n\in\mathbb{Z}} \left|\frac{\bar{z} - 1 + i - i\zeta + i\Omega_{m,n}}{z - 1 - i + i\zeta - i\Omega_{m,n}}\right|^2 = \prod_{m,n\in\mathbb{Z}} \left|\frac{z - 1 - i + i\bar{\zeta} - i\overline{\Omega_{m,n}}}{z - \zeta_2 - 2 - \Omega_{-n,m}}\right|^2$$

$$= \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \zeta_1 - \Omega_{n,m}}{z - \zeta_2 - \Omega_{-n+1,m}}\right|^2$$

where

$$-1 - i + i\bar{\zeta} - i\overline{\Omega_{m,n}} = -(1 + i - i\bar{\zeta}) - i\overline{(2m + 2ni)} = -\zeta_1 - (2n + 2mi)$$

$$= -\zeta_1 - \Omega_{n,m}$$

$$z - 1 - i + i\zeta - i\Omega_{m,n} = z - (1 + i - i\zeta) - i(2m + 2ni)$$

$$= z - (2 - 2 + 1 + i - i\zeta) - (-2n + 2mi) = z - (2 + \zeta_2) - (-2n + 2mi)$$

$$= z - \zeta_2 - (2 - 2n + 2mi) = z - \zeta_2 - \Omega_{-n+1,m}$$

second, we have

$$\prod_{m,n\in\mathbb{Z}} \left|\frac{z_2 - \bar{\zeta} + \overline{\Omega_{m,n}}}{\bar{z_2} - \zeta + \Omega_{m,n}}\right|^2 = \prod_{m,n\in\mathbb{Z}} \left|\frac{-1 + i - iz - \bar{\zeta} + \overline{\Omega_{m,n}}}{-1 + i - iz - \zeta + \Omega_{m,n}}\right|^2$$

$$= \prod_{m,n\in\mathbb{Z}} \left|\frac{z - 1 - i - i\bar{\zeta} + i\overline{\Omega_{m,n}}}{z - 1 - i - i\zeta + i\Omega_{m,n}}\right|^2 = \prod_{m,n\in\mathbb{Z}} \left|\frac{z - (\bar{\zeta_2} + 2 + 2i) + i\overline{\Omega_{m,n}}}{z - (\bar{\zeta_1} + 2i) + i\Omega_{m,n}}\right|^2$$

$$= \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \bar{\zeta_2} - (2 + 2i - 2mi - 2n)}{z - \bar{\zeta_1} - (2i + 2n - 2mi)}\right|^2 = \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \bar{\zeta_2} - \Omega_{-n+1,-m+1}}{z - \bar{\zeta_1} - \Omega_{n,-m+1}}\right|^2$$

Third, we have

$$\prod_{m,n\in\mathbb{Z}} \left|\frac{z_3 - \zeta + \Omega_{m,n}}{\bar{z_3} - \zeta + \Omega_{m,n}}\right|^2 = \prod_{m,n\in\mathbb{Z}} \left|\frac{-\bar{z} - \zeta + \Omega_{m,n}}{-z - \zeta + \Omega_{m,n}}\right|^2$$

$$= \prod_{m,n\in\mathbb{Z}} \left|\frac{z + \bar{\zeta} - \overline{\Omega_{m,n}}}{z + \zeta - \Omega_{m,n}}\right|^2 = \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \zeta_3 - \Omega_{m,-n}}{z - \bar{\zeta_3} - \Omega_{m,n}}\right|^2$$

Thus

$$G_1(z,\zeta) = \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \bar{\zeta} - \Omega_{-m,n}}{z - \zeta - \Omega_{-m,-n}} \cdot \frac{z - \zeta_1 - \Omega_{n,m}}{z - \zeta_2 - \Omega_{-n+1,m}} \cdot \frac{z - \bar{\zeta_2} - \Omega_{-n+1,-m+1}}{z - \bar{\zeta_1} - \Omega_{n,-m+1}} \cdot \frac{z - \zeta_3 - \Omega_{m,-n}}{z - \bar{\zeta_3} - \Omega_{m,n}}\right|^2.$$

Multiplying by the following double products

$$\prod_{m,n\in\mathbb{Z}} \left|\frac{z - \bar{\zeta} - \Omega_{m,n}}{z - \bar{\zeta} - \Omega_{-m,n}}\right|^2 = 1\ ,\ \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \zeta - \Omega_{-m,-n}}{z - \zeta - \Omega_{m,n}}\right|^2 = 1$$

$$\prod_{m,n\in\mathbb{Z}} \left|\frac{z - \zeta_1 - \Omega_{m,n}}{z - \zeta_1 - \Omega_{n,m}}\right|^2 = 1\ ,\ \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \zeta_2 - \Omega_{-n+1,m}}{z - \zeta_2 - \Omega_{m,n}}\right|^2 = 1$$

$$\prod_{m,n\in\mathbb{Z}} \left|\frac{z - \bar{\zeta_2} - \Omega_{m,n}}{z - \bar{\zeta_2} - \Omega_{-n+1,-m+1}}\right|^2 = 1\ ,\ \prod_{m,n\in\mathbb{Z}} \left|\frac{z - \bar{\zeta_1} - \Omega_{n,-m+1}}{z - \bar{\zeta_1} - \Omega_{m,n}}\right|^2 = 1$$

$$\prod_{m,n\in\mathbb{Z}} \left|\frac{z - \zeta_3 - \Omega_{m,n}}{z - \zeta_3 - \Omega_{m,-n}}\right|^2 = 1$$

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We obtain

$$G_1(z,\zeta) = \prod_{m,n\in\mathbb{Z}} \left| \frac{z-\bar{\zeta}-\Omega_{m,n}}{z-\zeta-\Omega_{m,n}} \cdot \frac{z-\zeta_1-\Omega_{m,n}}{z-\overline{\zeta_1}-\Omega_{m,n}} \cdot \frac{z-\overline{\zeta_2}-\Omega_{m,n}}{z-\zeta_2-\Omega_{m,n}} \cdot \frac{z-\zeta_3-\Omega_{m,n}}{z-\overline{\zeta_3}-\Omega_{m,n}} \right|^2 = G_1(\zeta,z).$$

Theorem (2):

The Poisson kernel for T is given as

$$P(z,\zeta) = Re \sum_{m,n\in\mathbb{Z}} \begin{cases} 8i\left[\frac{1}{z-\zeta-\Omega_{m,n}} - \frac{1}{z-\zeta_1-\Omega_{m,n}} + \frac{1}{z-\zeta_2-\Omega_{m,n}}\right] & on\ \partial_1 T \\ 4\sqrt{2}(1+i)\left[\frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z-\zeta-\Omega_{m,n}} - \frac{1}{z-\overline{\zeta_3}-\Omega_{m,n}} + \frac{1}{z-\zeta_3-\Omega_{m,n}}\right] & on\ \partial_2 T \\ 8\left[\frac{1}{z-\zeta-\Omega_{m,n}} - \frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z-\zeta_1-\Omega_{m,n}} - \frac{1}{z-\overline{\zeta_2}-\Omega_{m,n}}\right] & on\ \partial_3 T \end{cases}.$$

Proof:

i. On $\partial_1 T$ we have $\partial_\nu G_1(z,\zeta) = -2Re(i\partial_z)G_1(z,\zeta)$

$$z = \bar{z} \quad , \quad z_2 = z_1 - 2 \quad , \quad z_3 = -z\ .$$

Hence

$$\partial_\nu G_1(z,\zeta) = -4Re \sum_{m,n\in\mathbb{Z}} \left[\frac{i}{\zeta-z-\Omega_{m,n}} - \frac{i}{\bar{\zeta}-z-\overline{\Omega_{m,n}}} + \frac{1}{\overline{\zeta-z_1-\Omega_{m,n}}} - \frac{1}{\zeta-\overline{z_1}-\Omega_{m,n}} + + \frac{1}{\zeta-z_2-\Omega_{m,n}} - \frac{1}{\bar{\zeta}-z_2-\overline{\Omega_{m,n}}} + \frac{i}{\overline{\zeta-z_3-\Omega_{m,n}}} - \frac{i}{\zeta-\overline{z_3}-\Omega_{m,n}}\right].$$

$$= -4Re \sum_{m,n\in\mathbb{Z}} \left[\frac{2i}{\zeta-z-\Omega_{m,n}} + \frac{1}{\overline{\zeta-z_1-\Omega_{m,n}}} - \frac{1}{\zeta-\overline{z_1}-\Omega_{m,n}} + \frac{1}{\zeta-z_2-\Omega_{m,n}} - \frac{1}{\bar{\zeta}-z_2-\overline{\Omega_{m,n}}}\right].$$

rewriting the last four terms as

$$\frac{1}{\overline{\zeta-z_1-\Omega_{m,n}}} - \frac{1}{\zeta-\overline{z_1}-\Omega_{m,n}} = \frac{i}{z-\zeta_1-\Omega_{n,-m}} - \frac{i}{z-\zeta_2-\Omega_{-n+1,m}}.$$

$$\frac{1}{\zeta-z_2-\Omega_{m,n}} - \frac{1}{\bar{\zeta}-z_2-\overline{\Omega_{m,n}}} = \frac{i}{z-\overline{\zeta_2}-\Omega_{-n+1,-m+1}} - \frac{i}{z-\overline{\zeta_1}-\Omega_{n,-m+1}}.$$

where $\zeta_1 = -i\bar{\zeta} + 1 + i \quad , \quad \zeta_2 = -i\zeta - 1 + i\ .$

That's gives

$$\partial_\nu G_1(z,\zeta) = -8Re \sum_{m,n\in\mathbb{Z}} i\left[-\frac{1}{z-\zeta-\Omega_{m,n}} + \frac{1}{z-\zeta_1-\Omega_{m,n}} - \frac{1}{z-\zeta_2-\Omega_{m,n}}\right].$$

ii. On $\partial_2 T$ we have $\partial_\nu G_1(z,\zeta) = \left[\left(\frac{1+i}{\sqrt{2}}\right)\partial_z + \left(\frac{1-i}{\sqrt{2}}\right)\partial_{\bar{z}}\right] G_1(z,\zeta)$

$z = z_1 \quad , \quad z_2 = z_3$

$$\partial_z G_1(z,\zeta) = 2\sum_{m,n\in\mathbb{Z}} \left[-\frac{1}{\bar{\zeta}-z-\overline{\Omega_{m,n}}} + \frac{1}{\zeta-z-\Omega_{m,n}} - \frac{i}{\overline{\zeta-z_1-\Omega_{m,n}}} + \frac{i}{\zeta-\overline{z_1}-\Omega_{m,n}} - \frac{i}{\zeta-z_2-\Omega_{m,n}} + + \frac{i}{\bar{\zeta}-z_2-\overline{\Omega_{m,n}}} + \frac{1}{\overline{\zeta-z_3-\Omega_{m,n}}} - \frac{1}{\zeta-\overline{z_3}-\Omega_{m,n}}\right].$$

Hence

$$\left(\frac{1+i}{\sqrt{2}}\right)\partial_z G_1(z,\zeta) = \sqrt{2}\sum_{m,n\in\mathbb{Z}} \left[\frac{1+i}{z-\bar{\zeta}-\overline{\Omega_{m,n}}} + \frac{1-i}{\bar{z}-\zeta-\Omega_{m,n}} + \frac{-1+i}{\overline{z-\zeta-\Omega_{m,n}}} + \frac{-1-i}{z-\zeta-\Omega_{m,n}} + + \frac{1-i}{\bar{z}+\zeta-\Omega_{m,n}} + \frac{1+i}{z+\bar{\zeta}-\overline{\Omega_{m,n}}} + \frac{-1-i}{z+\zeta-\Omega_{m,n}} + \frac{-1+i}{\overline{z+\zeta-\Omega_{m,n}}}\right].$$

$$= 2\sqrt{2}Re \sum_{m,n\in\mathbb{Z}} (1+i)\left[\frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z-\zeta-\Omega_{m,n}} + \frac{1}{z+\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z+\zeta-\Omega_{m,n}}\right].$$

That's gives

$$\left(\frac{1+i}{\sqrt{2}}\right)\partial_z G_1(z,\zeta) = 2\sqrt{2}Re\sum_{m,n\in\mathbb{Z}}(1+i)\left[\frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z-\zeta-\Omega_{m,n}} + \frac{1}{z-\zeta_3-\Omega_{m,n}} - -\frac{1}{z-\overline{\zeta_3}-\Omega_{m,n}}\right].$$

where $\zeta_3 = -\bar{\zeta}$.

Similarly

$$\left(\frac{1-i}{\sqrt{2}}\right)\partial_{\bar{z}} G_1(z,\zeta) = 2\sqrt{2}Re\sum_{m,n\in\mathbb{Z}}(1+i)\left[\frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z-\zeta-\Omega_{m,n}} + \frac{1}{z-\zeta_3-\Omega_{m,n}} - -\frac{1}{z-\overline{\zeta_3}-\Omega_{m,n}}\right].$$

So we obtain

$$\partial_v G_1(z,\zeta) = \left[\left(\frac{1+i}{\sqrt{2}}\right)\partial_z + \left(\frac{1-i}{\sqrt{2}}\right)\partial_{\bar{z}}\right]G_1(z,\zeta) = 4\sqrt{2}Re\sum_{m,n\in\mathbb{Z}}(1+i)\left[\frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - -\frac{1}{z-\zeta-\Omega_{m,n}} + \frac{1}{z-\zeta_3-\Omega_{m,n}} - \frac{1}{z-\overline{\zeta_3}-\Omega_{m,n}}\right].$$

iii. On $\partial_3 T$ we have $\partial_v G_1(z,\zeta) = -2Re(\partial_z)G_1(z,\zeta)$

$z = z_3$

$$\partial_v G_1(z,\zeta) = -4Re\sum_{m,n\in\mathbb{Z}}\left|-\frac{1}{\bar{\zeta}-z-\overline{\Omega_{m,n}}} + \frac{1}{\zeta-z-\Omega_{m,n}} - \frac{i}{\bar{\zeta}-z_1-\overline{\Omega_{m,n}}} + \frac{i}{\zeta-\overline{z_1}-\Omega_{m,n}} + +\frac{i}{\bar{\zeta}-z_2-\overline{\Omega_{m,n}}} - \frac{i}{\zeta-z_2-\Omega_{m,n}} + \frac{1}{\overline{\zeta-z_3-\Omega_{m,n}}} - \frac{1}{\zeta-\overline{z_3}-\Omega_{m,n}}\right|^2.$$

Rewriting

$$-\frac{i}{\bar{\zeta}-z_1-\overline{\Omega_{m,n}}} + \frac{i}{\zeta-\overline{z_1}-\Omega_{m,n}} = -\frac{i}{-1+i-iz+\bar{\zeta}-\overline{\Omega_{m,n}}} + \frac{i}{-1+i-iz+\zeta-\Omega_{m,n}} = \frac{1}{z-\zeta_1-\Omega_{n,m}} + +\frac{1}{\bar{z}-\overline{\zeta_1}-\Omega_{m,n}}.$$

$$\frac{i}{\bar{\zeta}-z_2-\overline{\Omega_{m,n}}} - \frac{i}{\zeta-z_2-\Omega_{m,n}} = \frac{i}{iz+1-i+\bar{\zeta}-\overline{\Omega_{m,n}}} - \frac{i}{iz+1-i+\zeta-\Omega_{m,n}} = \frac{1}{z-\overline{\zeta_2}-\Omega_{-n+1,-m+1}} + +\frac{1}{\bar{z}-\zeta_2-\Omega_{-n,m-1}}.$$

where $\zeta_1 = -i\bar{\zeta} + 1 + i$, $\zeta_2 = -i\zeta - 1 + i$

Hence

$$\partial_v G_1(z,\zeta) = -8Re\sum_{m,n\in\mathbb{Z}}\left|\frac{1}{z-\bar{\zeta}-\Omega_{m,n}} - \frac{1}{z-\zeta-\Omega_{m,n}} + \frac{1}{z-\zeta_1-\Omega_{m,n}} + \frac{1}{z-\overline{\zeta_2}-\Omega_{m,n}}\right|^2.$$

Noticing that

$$z_2 = -\overline{z_1} \quad , \quad z_3 = -\bar{z}.$$

We can rewriting the function $G_1(z,\zeta)$ as

$$G_1(z,\zeta) = \prod_{m,n\in\mathbb{Z}}\left|\frac{\left[\left(\zeta-\Omega_{m,n}\right)^2-\bar{z}^2\right]^2.\left[\left(\zeta-\Omega_{m,n}\right)^2-z_1^2\right]^2}{\left[\left(\zeta-\Omega_{m,n}\right)^2-z^2\right]^2.\left[\left(\zeta-\Omega_{m,n}\right)^2-\bar{z}_1^2\right]^2}\right|^2.$$

First, we have

$$\left[\left(\zeta-\Omega_{m,n}\right)^2-\bar{z}^2\right]^2.\left[\left(\zeta-\Omega_{m,n}\right)^2-z_1^2\right]^2 = \left(\zeta-\Omega_{m,n}\right)^4 - (\bar{z}^2+z_1^2)\left(\zeta-\Omega_{m,n}\right)^2 + +(\bar{z}.z_1)^2. \qquad (*)$$

on the other hand, we have

$$\bar{z}^2 + z_1^2 = 2(z_1 + \bar{z} - 1).$$

Hence we can rewrite (*)

$$\left(\zeta-\Omega_{m,n}\right)^4 - 2(z_1+\bar{z}-1)\left(\zeta-\Omega_{m,n}\right)^2 + (\bar{z}.z_1)^2 = \left[\left(\zeta-\Omega_{m,n}\right)^2 - z_1 - \bar{z} + 1\right]^2 + +(\bar{z}.z_1)^2 - (z_1+\bar{z}-1)^2. \; (**)$$

Fixing (**) we obtain

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$$\left[\left(\zeta-\Omega_{m,n}\right)^2-(1-i)\bar{z}-i\right]^2-(\bar{z}+i)^2(\bar{z}-1)^2.$$

Similarly, we have

$$\left[\left(\zeta-\Omega_{m,n}\right)^2-z^2\right]^2.\left[\left(\zeta-\Omega_{m,n}\right)^2-\bar{z}_1^2\right]^2=\left[\left(\zeta-\Omega_{m,n}\right)^2-(1+i)z+i\right]^2--(z-i)^2(z-1)^2.$$

Finally, the harmonic green function for T is represented as the following

$$G_1(z,\zeta)=\prod_{m,n\in\mathbb{Z}}\left|\frac{\left[\left(\bar{\zeta}-\Omega_{m,n}\right)^2-(1+i)z+i\right]^2-(z-i)^2(z-1)^2}{\left[\left(\zeta-\Omega_{m,n}\right)^2-(1+i)z+i\right]^2-(z-i)^2(z-1)^2}\right|^2.$$

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