Novelty Journals

ISSN 2394-9651

International Journal of Novel Research in Physics Chemistry & Mathematics
Vol. 7, Issue 3, pp: (1-4), Month: September - December 2020, Available at: www.noveltyjournals.com

Some Fractional Differential Formulas

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***Abstract*: This paper studies the fractional differential problem. The Jumarie's modified Riemann-Liouville (R-L) fractional derivative and a new multiplication are introduced and we use Leibniz rule for fractional derivative and the fractional differential properties of several fractional functions to obtain three fractional differential formulas. Moreover, our results are generalizations of classical differential calculus.**

***Keywords*: Fractional differential problem, Jumarie's modified R-L fractional derivative, New multiplication, Leibniz rule for fractional derivative, Fractional differential formulas.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis which involves the investigation and applications of integrals and derivatives of arbitrary order. Although fractional calculus has almost the same long history as the classical calculus, it was only in recent decades that its theory and applications have rapidly developed. Oldham and Spanier [1] published the first monograph in 1974. Ross [2] edited the first proceedings that was published in 1975. Thereafter theory and applications of fractional calculus have attracted much interest and have become a vibrant research area. Nowadays, the number of monographs and proceedings devoted to fractional calculus has reached several dozen, e.g. [3–11]. Fractional calculus is not like the traditional calculus, there is no unique definition of fractional derivation and integration. The commonly used definitions are the Riemann-Liouville (R-L) fractional derivative [5], the Caputo definition of fractional derivative [5], the Grunwald-Letinikov (G-L) fractional derivative [5], and the Jumarie's modified R-L fractional derivative [12].

In this paper, Leibniz rule for fractional derivative is used to obtain three fractional differential formulas. We introduce a new multiplication and take advantage of the fractional differential properties of some elementary fractional functions such as fractional exponential function, fractional sine and cosine functions, regarding the Jumarie type of modified R-L fractional derivatives. Furthermore, our results are generalizations of traditional differential calculus.

II. PRELIMINARIES

Firstly, the fractional differentiation adopted in this article is introduced below.

Notation 2.1: If α is a real number, then

$$[\alpha]=\begin{cases}0 & \text{if } \alpha<0,\\ \text{the greatest integer less than or equal to } \alpha & \text{if } \alpha\geq 0.\end{cases} \tag{1}$$

Definition 2.2: Let α be a real number, m be a positive integer, and $f(x)\in C^{[\alpha]}([a,b])$. The modified Riemann-Liouville fractional derivatives of Jumarie type ([13]) is defined by

$$ {}_aD_x^{\alpha}[f(x)]=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_a^x(x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x(x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\leq\alpha<1\\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m\leq\alpha<m+1\end{cases} \tag{2}$$

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where $\Gamma(y)=\int_0^{\infty} t^{y-1}e^{-t}dt$ is the gamma function defined on $y>0$, and $\left({}_aD_x^{\alpha}\right)^n=\left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)$ is the n-th order fractional derivative of ${}_aD_x^{\alpha}$. We note that $\left({}_aD_x^{\alpha}\right)^n \neq {}_aD_x^{n\alpha}$ in general, and we have the following property [14].

Proposition 2.3: *Let α,β,c be real numbers and $\beta\geq\alpha>0$, then*

$$ {}_0D_x^{\alpha}\left[x^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{3}$$

and

$$ {}_0D_x^{\alpha}[c]=0. \tag{4}$$

In the following, we introduce some fractional functions and their properties.

Definition 2.4 ([15])**:** The Mittag-Leffler function is defined by

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{5}$$

where α is a real number, $\alpha\geq 0$, and z is a complex number.

The Mittag-Leffler function has gained importance and popularity during the last one and a half decades due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences.

Definition 2.5 ([16])**:** Let $0<\alpha\leq 1$, λ be a complex number, and x be a real variable. $E_{\alpha}(\lambda x^{\alpha})$ is called α-order fractional exponential function and the α-order fractional cosine and sine function are defined as follows:

$$cos_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{6}$$

and

$$sin_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+1}x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{7}$$

Remark 2.6: If $\alpha=1,\lambda=1$, then $cos_1(x)=cosx$, and $sin_1(x)=sinx$.

Notation 2.7: Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, and a,b are real numbers. a, the real part of z, is denoted by $\mathrm{Re}(z)$; b, the imaginary part of z, is denoted by $\mathrm{Im}(z)$.

Proposition 2.8 (fractional Euler's formula): *Let* $0<\alpha\leq 1$, *then*

$$E_{\alpha}(ix^{\alpha})=cos_{\alpha}(x^{\alpha})+isin_{\alpha}(x^{\alpha}). \tag{8}$$

In the following, we define a new multiplication of fractional functions.

Definition 2.9: Let λ,μ,z be complex numbers, $0<\alpha\leq 1$, j,l be non-negative integers, and a_k,b_k be real numbers, $p_r(z)=\begin{cases}\frac{1}{\Gamma(r\alpha+1)}z^r & if\ r\geq 0,\\ 0 & if\ r<0.\end{cases}$ for all integers r. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^{\alpha})\otimes p_l(\mu y^{\alpha})=\frac{1}{\Gamma(j\alpha+1)}(\lambda x^{\alpha})^j\otimes\frac{1}{\Gamma(l\alpha+1)}(\mu y^{\alpha})^l=\frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^{\alpha})^j(\mu y^{\alpha})^l, \tag{9}$$

where $\binom{j+l}{j}=\frac{(j+l)!}{j!l!}$.

If $f_{\alpha}(\lambda x^{\alpha})$ and $g_{\alpha}(\mu y^{\alpha})$ are two fractional functions,

$$f_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}a_k\,p_k(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^{\alpha})^k, \tag{10}$$

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$$g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\mu y^\alpha)^k, \tag{11}$$

then we define

$$f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu y^\alpha)\right). \tag{12}$$

Proposition 2.10: $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda x^\alpha)^{k-m} (\mu y^\alpha)^m.$ (13)

Definition 2.11: Let $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes n} = f_\alpha(\lambda x^\alpha) \otimes \cdots \otimes f_\alpha(\lambda x^\alpha)$ be the n times $\otimes$ product of the fractional function $f_\alpha(\lambda x^\alpha)$. If $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\lambda x^\alpha) = 1$, then $g_\alpha(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^\alpha)$, and is denoted by $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes -1}$.

Remark 2.12: The $\otimes$ multiplication satisfies the commutative law and the associate law, and is the generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha = 1$.

III. METHODS AND RESULTS

In the following, we give some properties of fractional functions discussed in this article.

Proposition 3.1 ([17])**:** *Let* $0 < \alpha \le 1$, *q be a non-negative integer, and j be an integer. Then*

$$\left({}_0D_x^\alpha\right)^q \left[p_j(x^\alpha)\right] = p_{j-q}(x^\alpha). \tag{14}$$

Proposition 3.2 ([17])**:** *Let* $0 < \alpha \le 1$, *k be a non-negative integer, and c be a real number, then*

$$\left({}_0D_x^\alpha\right)^k \left[E_\alpha(cx^\alpha)\right] = c^k E_\alpha(cx^\alpha). \tag{15}$$

Proposition 3.3 ([17])**:** *Assume that* $0 < \alpha \le 1$, *k is a non-negative integer, and b is a real number, then*

$$\left({}_0D_x^\alpha\right)^k \left[sin_\alpha(bx^\alpha)\right] = b^k \left[cos\frac{k\pi}{2} \cdot sin_\alpha(bx^\alpha) + sin\frac{k\pi}{2} \cdot cos_\alpha(bx^\alpha)\right], \tag{16}$$

and

$$\left({}_0D_x^\alpha\right)^k \left[cos_\alpha(bx^\alpha)\right] = b^k \left[cos\frac{k\pi}{2} \cdot cos_\alpha(bx^\alpha) - sin\frac{k\pi}{2} \cdot sin_\alpha(bx^\alpha)\right]. \tag{17}$$

Theorem 3.4 (Leibniz rule for fractional derivatives): *Let f_α, g_α be fractional functions, λ, μ be complex numbers, and n be a positive integer, then*

$$\left({}_0D_x^\alpha\right)^n \left[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu x^\alpha)\right] = \sum_{k=0}^{n} \binom{n}{k} \left({}_0D_x^\alpha\right)^{n-k} \left[f_\alpha(\lambda x^\alpha)\right] \otimes \left({}_0D_x^\alpha\right)^k \left[g_\alpha(\mu x^\alpha)\right] \tag{18}$$

The following is the major result of this paper, we provide three fractional differential formulas.

Theorem 3.5: *Suppose that* $0 < \alpha \le 1$, *n, j are positive integers, and b, c are real numbers, then*

$$\left({}_0D_x^\alpha\right)^n \left[p_j(x^\alpha) \otimes E_\alpha(cx^\alpha)\right] = \sum_{k=0}^{n} \binom{n}{k} c^k \cdot p_{j-n+k}(x^\alpha) \otimes E_\alpha(cx^\alpha), \tag{19}$$

$$\left({}_0D_x^\alpha\right)^n \left[p_j(x^\alpha) \otimes sin_\alpha(bx^\alpha)\right] = \sum_{k=0}^{n} \binom{n}{k} b^k \cdot p_{j-n+k}(x^\alpha) \otimes \left[cos\frac{k\pi}{2} \cdot sin_\alpha(bx^\alpha) + sin\frac{k\pi}{2} \cdot cos_\alpha(bx^\alpha)\right], \tag{20}$$

$$\left({}_0D_x^\alpha\right)^n \left[p_j(x^\alpha) \otimes cos_\alpha(bx^\alpha)\right] = \sum_{k=0}^{n} \binom{n}{k} b^k \cdot p_{j-n+k}(x^\alpha) \otimes \left[cos\frac{k\pi}{2} \cdot cos_\alpha(bx^\alpha) - sin\frac{k\pi}{2} \cdot sin_\alpha(bx^\alpha)\right]. \tag{21}$$

Proof Using Proposition 3.1, 3.2, 3.3 and Theorem 3.4 yields

$$\left({}_0D_x^\alpha\right)^n \left[p_j(x^\alpha) \otimes E_\alpha(cx^\alpha)\right] = \sum_{k=0}^{n} \binom{n}{k} \left({}_0D_x^\alpha\right)^{n-k} \left[p_j(x^\alpha)\right] \otimes \left({}_0D_x^\alpha\right)^k \left[E_\alpha(cx^\alpha)\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} c^k \cdot p_{j-n+k}(x^\alpha) \otimes E_\alpha(cx^\alpha).$$

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And

$$\left({}_0D_x^{\alpha}\right)^n\left[p_j(x^{\alpha})\otimes sin_{\alpha}(bx^{\alpha})\right]$$
$$=\sum_{k=0}^{n}\binom{n}{k}\left({}_0D_x^{\alpha}\right)^{n-k}\left[p_j(x^{\alpha})\right]\otimes\left({}_0D_x^{\alpha}\right)^{k}\left[sin_{\alpha}(bx^{\alpha})\right]$$
$$=\sum_{k=0}^{n}\binom{n}{k}\left({}_0D_x^{\alpha}\right)^{n-k}\left[p_j(x^{\alpha})\right]\otimes b^k\left[cos\frac{k\pi}{2}\cdot sin_{\alpha}(bx^{\alpha})+sin\frac{k\pi}{2}\cdot cos_{\alpha}(bx^{\alpha})\right]$$
$$=\sum_{k=0}^{n}\binom{n}{k}b^k\cdot p_{j-n+k}(x^{\alpha})\otimes\left[cos\frac{k\pi}{2}\cdot sin_{\alpha}(bx^{\alpha})+sin\frac{k\pi}{2}\cdot cos_{\alpha}(bx^{\alpha})\right].$$

Similarly, we can easily obtain Eq. (21). Q.e.d.

IV. CONCLUSION

As mentioned above, Leibniz rule for fractional derivatives and some fractional differential properties play important roles in this study. On the other hand, the three fractional differential formulas discussed in this paper are generalizations of classical differential calculus. In fact, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of engineering mathematics.

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