Integral Form of Particular Solution of Non-homogeneous Linear Fractional Differential Equation with Constant Coefficients

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Abstract: In this study, we use product rule of fractional functions to obtain the integral form of particular solution of non-homogeneous linear fractional differential equation (FDE) with constant coefficients, regarding Jumarie’s modified Riemann-Liouville (R-L) fractional derivative. On the other hand, several examples are proposed for demonstrating the advantage of our method.

Keywords: Product rule, Integral form, Particular solution, Non-homogeneous linear FDE with constant coefficients, Jumarie’s modified R-L fractional derivative.

I. INTRODUCTION

Fractional differential equations (FDEs) occur in numerous complex systems in life science such as rheology, viscoelasticity, porous media, electrochemistry, electromagnetism, dynamics of earthquakes, geology, viscoelastic materials, bioengineering, signal processing, optics, biosciences, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, converters, electromagnetic waves, and many other scientific areas [1-11]. Fractional calculus is not like the traditional calculus, there is no unique definition of fractional derivation and integration. The commonly used definitions are the Riemann–Liouville (R-L) fractional derivative [12], the Caputo definition of fractional derivative [12], the Grunwald-Letnikov (G-L) fractional derivative [12], and the Jumarie’s modified R-L fractional derivative [13].

The differential equations in different form of fractional derivatives give different type of solutions. Therefore, there is no standard methods to solve FDEs. Ghosh et al. [15] developed analytical method for solution of linear fractional differential equations with Jumarie type of modified R-L derivative. The aim of this article is to obtain the integral form of particular solution of non-homogeneous linear FDE with constant coefficients, regarding Jumarie’s modified R-L fractional derivative. The product rule of fractional functions plays an important role in this study. Moreover, the main result obtained in this paper is different from [16], and is the generalization of particular solution of non-homogeneous linear ordinary differential equations with constant coefficients. In addition, we propose some examples to demonstrate the validity of our results.

II. PRELIMINARIES

In the following, we introduce some fractional functions and their properties.

Definition 2.1 ([14]): If $\alpha > 0$, and $z$ is a complex variable. The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}$$

(1)
Definition 2.2 ([15]): Let $0 < \alpha \leq 1$, $\lambda$ be a complex number, and $t$ be a real variable, then $E_\alpha(\lambda t^\alpha)$ is called $\alpha$-order fractional exponential function, and the $\alpha$-order fractional cosine and sine function are defined as follows:

$$cos_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} t^{\frac{2k+1}{\alpha}}}{\Gamma(2k+1+\alpha)}.$$

and

$$sin_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} t^{\frac{2k+1}{\alpha}}}{\Gamma(2k+1+\alpha)}.$$

Notation 2.3: Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and $a, b$ are real numbers. $a$ the real part of $z$, denoted as $\text{Re}(z)$; $b$ the imaginary part of $z$, denoted as $\text{Im}(z)$.

Proposition 2.4 (fractional Euler’s formula) ([18]): If $0 < \alpha \leq 1$, then

$$E_\alpha(it^\alpha) = cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha).$$

Next, we define a new multiplication of fractional functions such that some properties, for instance, product rule and chain rule are correct.

Definition 2.5 ([17]): Suppose that $\lambda, \mu, z$ are complex numbers, $0 < \alpha \leq 1, j, l, k$ are non-negative integers, and $a_k, b_k$ are real numbers, $p_k(z) = \frac{1}{\Gamma(k+1)} z^k$ for all $k$. Then we define

$$p_j(\lambda t^\alpha) \Theta p_l(\mu s^\beta) = \frac{1}{\Gamma(j+1)} (\lambda t^\alpha)^{j} \Theta \frac{1}{\Gamma(l+1)} (\mu s^\beta)^{l}.$$

$$= \frac{1}{\Gamma((j+l)+1)} \left(\frac{j}{j+l}\right) (\lambda t^\alpha)^{j} (\mu s^\beta)^{l},$$

where \( \frac{j}{j+l} \). If $f_\alpha(\lambda t^\alpha)$ and $g_\alpha(\mu s^\beta)$ are two fractional functions,

$$f_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} a_k p_k(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k+1)} (\lambda t^\alpha)^k,$$

$$g_\alpha(\mu s^\beta) = \sum_{k=0}^{\infty} b_k p_k(\mu s^\beta) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k+1)} (\mu s^\beta)^k,$$

then we define

$$f_\alpha(\lambda t^\alpha) \Theta g_\alpha(\mu s^\beta) = \sum_{k=0}^{\infty} a_k p_k(\lambda t^\alpha) \Theta \sum_{k=0}^{\infty} b_k p_k(\mu s^\beta)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda t^\alpha) \Theta p_m(\mu s^\beta)\right).$$

Proposition 2.6 ([17]): $f_\alpha(\lambda t^\alpha) \Theta g_\alpha(\mu s^\beta) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda t^\alpha)^{k-m} (\mu s^\beta)^m.$

Definition 2.7: $(f_\alpha(\lambda t^\alpha))^\otimes n = f_\alpha(\lambda t^\alpha) \Theta \cdots \Theta f_\alpha(\lambda t^\alpha)$ is the $n$ times product of the fractional function $f_\alpha(\lambda t^\alpha).$ And $cos_\alpha^\otimes n(bt^\alpha) = (cos_\alpha(bt^\alpha))^\otimes n, sin_\alpha^\otimes n(bt^\alpha) = (sin_\alpha(bt^\alpha))^\otimes n.$

Remark 2.8: The $\Theta$ multiplication satisfies the commutative law and the associate law, and it is the generalization of traditional multiplication, since the $\Theta$ multiplication becomes the ordinary multiplication if $\alpha = 1.$

Proposition 2.9 ([17]): $E_\alpha(\lambda t^\alpha) \Theta E_\alpha(\mu s^\beta) = E_\alpha(\lambda t^\alpha + \mu s^\beta).$

Corollary 2.10: $E_\alpha(\lambda t^\alpha) \Theta E_\alpha(\mu t^\alpha) = E_\alpha(\lambda + \mu) t^\alpha.$

Remark 2.11: Peng and Li [20] give an example to show that $E_\alpha(\lambda t^\alpha) \cdot E_\alpha(\lambda s^\beta) = E_\alpha(\lambda(t + s)^\alpha)$ is not true for $0 < \alpha < 1.$ On the other hand, Area et al. [21] also provide a counterexample for $E_\alpha(\lambda t^\alpha) \cdot E_\alpha(\mu t^\alpha) = E_\alpha((\lambda + \mu) t^\alpha), 0 < \alpha < 1.$
**Proposition 2.12:** Assume that $0 < \alpha \leq 1$, and $a, b$ are real numbers, then

$$E_a(at^a) \otimes \cos_a(bt^a) = \frac{1}{2} [E_a((a + ib)t^a) + E_a((a - ib)t^a)],$$

(12)

and

$$E_a(at^a) \otimes \sin_a(bt^a) = \frac{1}{2i} [E_a((a + ib)t^a) - E_a((a - ib)t^a)].$$

(13)

**Proof**

Since

$$E_a((a + ib)t^a) = E_a(at^a) \otimes E_a(ibt^a) \quad \text{(by Eq. (11))}$$

$$= E_a(at^a) \otimes [\cos_a(bt^a) + isin_a(bt^a)] \quad \text{(by Eq. (4))}$$

$$= E_a(at^a) \otimes \cos_a(bt^a) + iE_a(at^a) \otimes isin_a(bt^a).$$

It follows that

$$E_a(at^a) \otimes \cos_a(bt^a) = \text{Re}[E_a((a + ib)t^a)] = \frac{1}{2} [E_a((a + ib)t^a) + E_a((a - ib)t^a)].$$

And

$$E_a(at^a) \otimes \sin_a(bt^a) = \text{Im}[E_a((a + ib)t^a)] = \frac{1}{2i} [E_a((a + ib)t^a) - E_a((a - ib)t^a)].$$

q.e.d.

**Proposition 2.13:** If $0 < \alpha \leq 1$, $b$ is a real number, and $m$ is a positive integer, then

$$\cos_a^{\otimes m}(bt^a) = \frac{1}{2^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \cos_a((m - 2l)bt^a),$$

(14)

and

$$\sin_a^{\otimes m}(bt^a) = \frac{1}{(-2)^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \left( \cos \frac{m \pi}{2} \cdot \cos_a((m - 2l)bt^a) - \sin \frac{m \pi}{2} \cdot \sin_a((m - 2l)bt^a) \right).$$

(15)

**Proof**

$$\cos_a^{\otimes m}(bt^a) = \left( \frac{1}{2} (E_a(ibt^a) + E_a(-ibt^a)) \right) \otimes \ldots \otimes \left( \frac{1}{2} (E_a(ibt^a) + E_a(-ibt^a)) \right)$$

$$= \frac{1}{2^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \left( E_a(ibt^a)^{\otimes m-l} \otimes E_a(-ibt^a)^{\otimes l} \right)$$

$$= \frac{1}{2^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} E_a((m - l)bt^a) \otimes E_a((-m - l)bt^a)$$

$$= \frac{1}{2^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} E_a(i(m - 2l)bt^a)$$

$$= \frac{1}{2^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \cos_a((m - 2l)bt^a). \quad \text{(by Eq. (4))}$$

And

$$\sin_a^{\otimes m}(bt^a)$$

$$= \left( \frac{1}{2} (E_a(ibt^a) - E_a(-ibt^a)) \right) \otimes \ldots \otimes \left( \frac{1}{2} (E_a(ibt^a) - E_a(-ibt^a)) \right)$$

$$= \frac{1}{(-2)^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \left( E_a(ibt^a)^{\otimes m-l} \otimes (-E_a(-ibt^a))^{\otimes l} \right)$$

$$= \frac{1}{(-2)^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} E_a(i(m - 2l)bt^a)$$

$$= \frac{1}{(-2)^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \left( \cos \frac{m \pi}{2} \cdot \cos_a((m - 2l)bt^a) + \sin \frac{m \pi}{2} \cdot \sin_a((m - 2l)bt^a) \right)$$

$$= \frac{1}{(-2)^m} \sum_{l=0}^{m \left( \frac{m-1}{2} \right)} \left( \cos \frac{m \pi}{2} \cdot \cos_a((m - 2l)bt^a) - \sin \frac{m \pi}{2} \cdot \sin_a((m - 2l)bt^a) \right).$$

q.e.d.
III. MAIN RESULTS

In this section, we will find the integral form of particular solution of non-homogeneous linear FDE with constant coefficients. Firstly, we introduce the fractional derivatives adopted in this paper.

**Notation 3.1:** If \( \alpha \) is a real number, we define

\[
[\alpha] = \begin{cases} 
0 & \text{if } \alpha < 0, \\
\text{the greatest integer less than or equal to } \alpha & \text{if } \alpha \geq 0.
\end{cases}
\]

**Definition 3.2:** Assume that \( \alpha \) is a real number, \( m \) is a positive integer, and \( f(t) \in C^{[\alpha]}([a, b]) \). The modified Riemann-Liouville fractional derivatives of Jumarie type ([13, 15]) is defined by

\[
(\ {}_aD_t^\alpha [f(t)] = \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \int_a^t (t - r)^{-\alpha-1} f(r) dr, & \text{if } \alpha < 0, \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t - r)^{-\alpha} [f(r) - f(a)] dr, & \text{if } 0 < \alpha < 1, \\
\frac{d^m}{dt^m} (\ {}_aD_t^{\alpha-m} [f(t)] , & \text{if } m \leq \alpha < m + 1
\end{cases}
\]

where \( \Gamma(y) = \int_0^\infty s^{y-1} e^{-s} ds \) is the gamma function defined on \( y > 0 \). For any positive integer \( n \), we define \( (\ {}_aD_t^\alpha )^n = (\ {}_aD_t^\beta )(\ {}_aD_t^\beta ) \ldots (\ {}_aD_t^\beta ) \), the \( n \)-th order fractional derivative of \( \ {}_aD_t^\alpha \). On the other hand, the fractional integral is defined by \( \{}_aI_t^\alpha f = \ {}_aI_t^{\alpha-m} \), where \( \alpha > 0 \). We have the following properties.

**Proposition 3.3** ([19]): Suppose that \( \alpha, \beta, c \) are real constants and \( 0 < \alpha \leq 1 \), then

\[
(\ {}_aD_t^\alpha )[t^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad \text{if } \beta \geq \alpha
\]

\[
(\ {}_aD_t^\alpha )[c] = 0,
\]

\[
(\ {}_aI_t^\alpha )[t^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}, \quad \text{if } \beta > -1.
\]

**Theorem 3.4** ([19]): If \( 0 < \alpha \leq 1 \) and \( f(t) \) is a continuous function, then

\[
(\ {}_aD_t^\alpha )[\int_a^t f(s) ds] = f(t).
\]

**Theorem 3.5:** If \( a, b, \alpha \) are real constants, \( a^2 + b^2 \neq 0 \), and \( 0 < \alpha \leq 1 \), then the fractional integrals

\[
(\ {}_aI_t^\alpha )[E_a(at^a) \otimes \cos_a(bt^a)] = \frac{1}{a^2 + b^2} E_a(at^a) \otimes (a \cos_a(bt^a) + b \sin_a(bt^a)) - \frac{a}{a^2 + b^2}.
\]

**Proof**

\[
(\ {}_aI_t^\alpha )[E_a(at^a) \otimes \cos_a(bt^a)]
\]

\[
= \left(\ {}_aI_t^\alpha \right) \left[\frac{1}{2} [E_a((a + ib)t^a) + E_a((a - ib)t^a)] \right] \quad \text{(by Eq. (11))}
\]

\[
= \frac{1}{2} \left(\ {}_aI_t^\alpha \right) [E_a((a + ib)t^a) + E_a((a - ib)t^a)]
\]

\[
= \frac{1}{2} \left[ \frac{1}{a + ib} E_a((a + ib)t^a) + \frac{1}{a - ib} E_a((a - ib)t^a) - \frac{1}{a + ib} - \frac{1}{a - ib} \right]
\]

\[
= \frac{1}{2(a^2 + b^2)} \left[ (a - ib)E_a((a + ib)t^a) + (a + ib)E_a((a - ib)t^a) - \frac{a}{a^2 + b^2} \right]
\]

\[
= \frac{1}{2(a^2 + b^2)} \left[ (a - ib) \left( E_a(at^a) \otimes (\cos_a(bt^a) + i \sin_a(bt^a)) \right) + (a + ib) \left( E_a(at^a) \otimes (\cos_a(bt^a) - i \sin_a(bt^a)) \right) \right]
\]

\[
- \frac{a}{a^2 + b^2}
\]

\[
= \frac{1}{a^2 + b^2} E_a(at^a) \otimes (a \cos_a(bt^a) + b \sin_a(bt^a)) - \frac{a}{a^2 + b^2}.
\]
On the other hand,
\[
\left( \frac{\partial}{\partial t^\alpha} \right)[E_a(at^\alpha) \otimes \sin_at^\alpha]
\]
\[
= \left( \frac{\partial}{\partial t^\alpha} \right) \left[ \frac{1}{2i} E_a((a+ib)t^\alpha) - E_a((-a-ib)t^\alpha) \right] \quad \text{(by Eq. (12))}
\]
\[
= \frac{1}{2i} \frac{\partial}{\partial t^\alpha} E_a((a+ib)t^\alpha) - \frac{1}{a+ib} E_a((-a-ib)t^\alpha) - \frac{1}{a+ib} + \frac{1}{a-ib}
\]
\[
= \frac{-1}{2(a^2+b^2)} \left( (b+ai)E_a((a+ib)t^\alpha) + (b-ai)E_a((-a+ib)t^\alpha) \right) + \frac{b}{a^2+b^2}
\]
\[
= \frac{-1}{a^2+b^2} E_a(at^\alpha) \otimes \left( \cos(bt^\alpha) - a \sin(bt^\alpha) \right) + \frac{b}{a^2+b^2}.
\]

The linearity property obviously holds by Definition 3.2.

\[
\left( \frac{\partial}{\partial t^\alpha} \right)[f_a(\lambda t^\alpha) + g_\alpha(\mu t^\alpha)] = a \left( \frac{\partial}{\partial t^\alpha} \right)[f_a(\lambda t^\alpha)] + b \left( \frac{\partial}{\partial t^\alpha} \right)[g_\alpha(\mu t^\alpha)].
\]

(23)

where \( f_a, g_\alpha \) are fractional functions, and \( a, b, \lambda, \mu \) are complex constants.

**Theorem 3.6 (product rule for fractional derivatives)** ([17]): Let \( 0 < \alpha \leq 1, \lambda, \mu \) be complex numbers, and \( f_a, g_\alpha \) be fractional functions, then

\[
\left( \frac{\partial}{\partial t^\alpha} \right)[f_a(\lambda t^\alpha) \otimes g_\alpha(\mu t^\alpha)] = \left( \frac{\partial}{\partial t^\alpha} \right)[f_a(\lambda t^\alpha)] \otimes g_\alpha(\mu t^\alpha) + f_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[g_\alpha(\mu t^\alpha)].
\]

(24)

The following is the differential form of particular solution of non-homogeneous linear FDE with constant coefficients.

**Theorem 3.7** ([18]): If \( 0 < \alpha \leq 1, n \) is a positive integer, \( a_0, a_1, \ldots, a_n \) are real constants, and \( a_n \neq 0 \). The non-homogeneous linear FDE with constant coefficients

\[
\left( a_n \left( \frac{\partial}{\partial t^\alpha} \right)^n + a_{n-1} \left( \frac{\partial}{\partial t^\alpha} \right)^{n-1} + \cdots + a_1 \left( \frac{\partial}{\partial t^\alpha} \right) + a_0 \right) [y(t^\alpha)] = r(t^\alpha)
\]

(25)

has the particular solution

\[
y_p(t^\alpha) = \left( \frac{1}{a_n \left( \frac{\partial}{\partial t^\alpha} \right)^n + a_{n-1} \left( \frac{\partial}{\partial t^\alpha} \right)^{n-1} + \cdots + a_1 \left( \frac{\partial}{\partial t^\alpha} \right) + a_0} \right) [r(t^\alpha)].
\]

(26)

To obtain the main result of this article, we need the following properties.

**Lemma 3.8:** Let \( \lambda \) be a complex number, then

\[
\left( \frac{1}{\frac{\partial}{\partial t^\alpha} - \lambda} \right) \left[ r(t^\alpha) \right] = E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)].
\]

(27)

**Proof**

\[
\left( \frac{\partial}{\partial t^\alpha} - \lambda \right) \left[ E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)] \right]
\]

\[
= \left( \frac{\partial}{\partial t^\alpha} \right) \left[ E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)] \right] - \lambda E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)]
\]

\[
= \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(\lambda t^\alpha)] \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)] + E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)]
\]

\[
- \lambda E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)] \quad \text{(by Theorem 3.6)}
\]

\[
= \lambda E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)] + r(t^\alpha) - \lambda E_a(\lambda t^\alpha) \otimes \left( \frac{\partial}{\partial t^\alpha} \right)[E_a(-\lambda t^\alpha) \otimes r(t^\alpha)]
\]

\[
r(t^\alpha).
\]

By Theorem 3.7, the desired result holds.

q.e.d.

**Theorem 3.9:** Let the assumptions be the same as Lemma 3.8, and \( \lambda_1, \lambda_2, \cdots, \lambda_n \) be the roots of the equation \( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \), where \( a_n \neq 0 \). Then

\[
\left( \frac{1}{a_n \left( \frac{\partial}{\partial t^\alpha} \right)^n + a_{n-1} \left( \frac{\partial}{\partial t^\alpha} \right)^{n-1} + \cdots + a_1 \left( \frac{\partial}{\partial t^\alpha} \right) + a_0} \right) [r(t^\alpha)]
\]

q.e.d.
Therefore, we can easily obtain the integral form of particular solution of non-homogeneous linear FDE with constant coefficients.

**Theorem 3.10:** The non-homogeneous linear FDE with constant coefficients

\[
\left(a_n \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right)^n + a_{n-1} \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right)^{n-1} + \cdots + a_1 \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right) + a_0 \right) y(t^\alpha) = r(t^\alpha) \tag{29}
\]

has the particular solution

\[
y_p(t^\alpha) = \frac{1}{a_n} \cdot \left[ E_a\left(\lambda_1 t^\alpha\right) \otimes \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right) \left(E_a\left((\lambda_1 - \lambda_n) t^\alpha\right) \otimes \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right) \cdots E_a\left((-\lambda_1 t^\alpha) \otimes r(t^\alpha)\right) \cdots \right] \tag{30}
\]

**Proof** By Theorem 3.7 and Theorem 3.9, the desired result holds. q.e.d.

**Remark 3.11:** The particular solution Eq. (30) may contain the general solution part of

\[
\left(a_n \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right)^n + a_{n-1} \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right)^{n-1} + \cdots + a_1 \left(\frac{D^{\alpha}}{\alpha^{\alpha}}\right) + a_0 \right) y(t^\alpha) = 0 \tag{31}
\]

So we can ignore the general solution part when we seek the particular solution Eq. (30), and the particular solution that the general solution part to be removed denoted as \( P\left(y_p(t^\alpha)\right) \).

### IV. EXAMPLES

In the following, we will give several examples to illustrate the integral form of particular solution of non-homogeneous linear FDE with constant coefficients.

**Example 4.1:** Let \(a, b\) be real numbers, and \(a \neq b\). The non-homogeneous linear FDE with constant coefficients

\[
\left(\left(\frac{D^{1/3}}{t^{1/3}}\right)^2 - 2a \left(\frac{D^{1/3}}{t^{1/3}}\right) + a^2\right) y\left(t^{1/3}\right) = E_{1/3}\left(bt^{1/3}\right) \tag{32}
\]

has the particular solution

\[
P\left(y_p\left(t^{1/3}\right)\right) = P\left(E_{1/3}\left(at^{1/3}\right) \otimes \left(\frac{D^{1/3}}{t^{1/3}}\right) \left[E_{1/3}\left((b-a)t^{1/3}\right)\right]\right)]
\]
Example 4.2: Let \( t \geq 0 \), then
\[
\left( \left( aD_t^{1/4} \right)^2 - 3 \left( aD_t^{1/4} \right) - 4 \right) y \left( t^{1/4} \right) = E_{1/4} \left( -5t^{1/4} \right) \otimes sin_{1/4} \left( 2t^{1/4} \right)
\]
has the particular solution
\[
P \left( y_p \left( t^{1/4} \right) \right) = P \left( E_{1/4} \left( 4t^{1/4} \right) \otimes \left( o_t^{1/4} \right) \left[ E_{1/4} \left( -5t^{1/4} \right) \otimes \left( o_t^{1/4} \right) \left[ E_{1/4} \left( -4t^{1/4} \right) \otimes sin_{1/4} \left( 2t^{1/4} \right) \right] \right] \right)
\]
\[
= P \left( E_{1/4} \left( 4t^{1/4} \right) \otimes \left( o_t^{1/4} \right) \left[ \frac{1}{20} E_{1/4} \left( -9t^{1/4} \right) \otimes \left( 2cos_{1/4} \left( 2t^{1/4} \right) + 4sin_{1/4} \left( 2t^{1/4} \right) \right) \right] \right) \quad \text{(by Theorem 3.5)}
\]
\[
= P \left( \frac{1}{20} E_{1/4} \left( 4t^{1/4} \right) \otimes \left( o_t^{1/4} \right) \left[ E_{1/4} \left( -9t^{1/4} \right) \otimes \left( 2cos_{1/4} \left( 2t^{1/4} \right) + 4sin_{1/4} \left( 2t^{1/4} \right) \right) \right] \right) \quad \text{(by Theorem 3.5)}
\]
\[
= E_{1/4} \left( -5t^{1/4} \right) \otimes \frac{26}{1700} cos_{1/4} \left( 2t^{1/4} \right) + \frac{32}{1700} sin_{1/4} \left( 2t^{1/4} \right). \quad (35)
\]

Example 4.3: \( \left( aD_t^{1/3} \right)^2 y \left( t^{1/3} \right) = cos_{1/3}^3 \left( 2t^{1/3} \right) \)
has the particular solution
\[
P \left( y_p \left( t^{1/3} \right) \right) = P \left( \left( o_t^{1/3} \right)^2 \left[ cos_{1/3}^3 \left( 2t^{1/3} \right) \right] \right)
\]
\[
= P \left( \left( o_t^{1/3} \right)^2 \left[ \frac{1}{4} cos_{1/3} \left( 6t^{1/3} \right) + \frac{3}{4} cos_{1/3} \left( 2t^{1/3} \right) \right] \right) \quad \text{(by Eq. (14))}
\]
\[
= P \left( \left( o_t^{1/3} \right) \left[ \frac{1}{24} sin_{1/3} \left( 6t^{1/3} \right) + \frac{3}{8} sin_{1/3} \left( 2t^{1/3} \right) \right] \right)
\]
\[
= -\frac{1}{144} cos_{1/3} \left( 6t^{1/3} \right) - \frac{2}{16} cos_{1/3} \left( 2t^{1/3} \right). \quad (37)
\]

Example 4.4: \( \left( aD_t^{1/5} + 1 \right) y \left( t^{1/5} \right) = t^{1/5} \)
has the particular solution
\[
P \left( y_p \left( t^{1/5} \right) \right) = P \left( E_{1/5} \left( -t^{1/5} \right) \otimes \left( o_t^{1/5} \right) \left[ E_{1/5} \left( t^{1/5} \right) \otimes t^{1/5} \right] \right)
\]
\[
= P \left( E_{1/5} \left( -t^{1/5} \right) \otimes \left( E_{1/5} \left( t^{1/5} \right) \otimes \left( t^{1/5} - \Gamma \left( \frac{6}{5} \right) \right) \right) \right)
\]
\[
= t^{1/5} - \Gamma \left( \frac{6}{5} \right). \quad (39)
\]

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V. CONCLUSIONS

The particular solution of non-homogeneous linear FDE with constant coefficients has two expressions: differential form and integral form. They are generalizations of particular solution of non-homogeneous linear ordinary differential equation with constant coefficients. In this paper, the integral form is proved by product rule of fractional functions. On the other hand, the differential form can be represented as formal Laurent series of fractional differential operator. In fact, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of engineering mathematics.

REFERENCES


