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Integral Form of Particular Solution of Non-homogeneous Linear Fractional Differential Equation with Constant Coefficients

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***Abstract*: In this study, we use product rule of fractional functions to obtain the integral form of particular solution of non-homogeneous linear fractional differential equation (FDE) with constant coefficients, regarding Jumarie's modified Riemann-Liouville (R-L) fractional derivative. On the other hand, several examples are proposed for demonstrating the advantage of our method.**

***Keywords*: Product rule, Integral form, Particular solution, Non-homogeneous linear FDE with constant coefficients, Jumarie's modified R-L fractional derivative.**

I. INTRODUCTION

Fractional differential equations (FDEs) occur in numerous complex systems in life science such as rheology, viscoelasticity, porous media, electrochemistry, electromagnetism, dynamics of earthquakes, geology, viscoelastic materials, bioengineering, signal processing, optics, biosciences, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, converters, electromagnetic waves, and many other scientific areas [1-11]. Fractional calculus is not like the traditional calculus, there is no unique definition of fractional derivation and integration. The commonly used definitions are the Riemann-Liouville (R-L) fractional derivative [12], the Caputo definition of fractional derivative [12], the Grunwald-Letinikov (G-L) fractional derivative [12], and the Jumarie's modified R-L fractional derivative [13].

The differential equations in different form of fractional derivatives give different type of solutions. Therefore, there is no standard methods to solve FDEs. Ghosh et al. [15] developed analytical method for solution of linear fractional differential equations with Jumarie type of modified R-L derivative. The aim of this article is to obtain the integral form of particular solution of non-homogeneous linear FDE with constant coefficients, regarding Jumarie's modified R-L fractional derivative. The product rule of fractional functions plays an important role in this study. Moreover, the main result obtained in this paper is different from [16], and is the generalization of particular solution of non-homogeneous linear ordinary differential equations with constant coefficients. In addition, we propose some examples to demonstrate the validity of our results.

II. PRELIMINARIES

In the following, we introduce some fractional functions and their properties.

Definition 2.1 ([14])**:** If $\alpha > 0$, and z is a complex variable. The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}. \quad (1)$$

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Definition 2.2 ([15]): Let $0 < \alpha \le 1$, λ be a complex number, and t be a real variable, then $E_\alpha(\lambda t^\alpha)$ is called α-order fractional exponential function, and the α-order fractional cosine and sine function are defined as follows:

$$cos_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} t^{2k\alpha}}{\Gamma(2k\alpha+1)}, \quad (2)$$

and

$$sin_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \quad (3)$$

Notation 2.3: Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a the real part of z, denoted as $\text{Re}(z)$; b the imaginary part of z, denoted as $\text{Im}(z)$.

Proposition 2.4 (fractional Euler's formula) ([18]): *If* $0 < \alpha \le 1$, *then*

$$E_\alpha(it^\alpha) = cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha). \quad (4)$$

Next, we define a new multiplication of fractional functions such that some properties, for instance, product rule and chain rule are correct.

Definition 2.5 ([17]): Suppose that λ, μ, z are complex numbers, $0 < \alpha \le 1$, j, l, k are non-negative integers, and a_k, b_k are real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)} z^k$ for all k. Then we define

$$\begin{aligned} p_j(\lambda t^\alpha) \otimes p_l(\mu s^\alpha) &= \frac{1}{\Gamma(j\alpha+1)} (\lambda t^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)} (\mu s^\alpha)^l \\ &= \frac{1}{\Gamma((j+l)\alpha+1)} \binom{j+l}{j} (\lambda t^\alpha)^j (\mu s^\alpha)^l, \end{aligned} \quad (5)$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

If $f_\alpha(\lambda t^\alpha)$ and $g_\alpha(\mu s^\alpha)$ are two fractional functions,

$$f_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\lambda t^\alpha)^k, \quad (6)$$

$$g_\alpha(\mu s^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu s^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\mu s^\alpha)^k, \quad (7)$$

then we define

$$\begin{aligned} f_\alpha(\lambda t^\alpha) \otimes g_\alpha(\mu s^\alpha) &= \sum_{k=0}^{\infty} a_k\, p_k(\lambda t^\alpha) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu s^\alpha) \\ &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda t^\alpha) \otimes p_m(\mu s^\alpha) \right). \end{aligned} \quad (8)$$

Proposition 2.6 ([17]): $f_\alpha(\lambda t^\alpha) \otimes g_\alpha(\mu s^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda t^\alpha)^{k-m} (\mu s^\alpha)^m. \quad (9)$

Definition 2.7: $(f_\alpha(\lambda t^\alpha))^{\otimes n} = f_\alpha(\lambda t^\alpha) \otimes \cdots \otimes f_\alpha(\lambda t^\alpha)$ is the n times product of the fractional function $f_\alpha(\lambda t^\alpha)$. And $cos_\alpha^{\otimes n}(bt^\alpha) = (cos_\alpha(bt^\alpha))^{\otimes n}$, $sin_\alpha^{\otimes n}(bt^\alpha) = (sin_\alpha(bt^\alpha))^{\otimes n}$.

Remark 2.8: The $\otimes$ multiplication satisfies the commutative law and the associate law, and it is the generalization of traditional multiplication, since the $\otimes$ multiplication becomes the ordinary multiplication if $\alpha = 1$.

Proposition 2.9 ([17]): $E_\alpha(\lambda t^\alpha) \otimes E_\alpha(\mu s^\alpha) = E_\alpha(\lambda t^\alpha + \mu s^\alpha). \quad (10)$

Corollary 2.10: $E_\alpha(\lambda t^\alpha) \otimes E_\alpha(\mu t^\alpha) = E_\alpha((\lambda + \mu) t^\alpha). \quad (11)$

Remark 2.11: Peng and Li [20] give an example to show that $E_\alpha(\lambda t^\alpha) \cdot E_\alpha(\lambda s^\alpha) = E_\alpha(\lambda (t+s)^\alpha)$ is not true for $0 < \alpha < 1$. On the other hand, Area *et al.* [21] also provide a counterexample for $E_\alpha(\lambda t^\alpha) \cdot E_\alpha(\mu t^\alpha) = E_\alpha((\lambda + \mu) t^\alpha)$, $0 < \alpha < 1$.

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Proposition 2.12: *Assume that* $0 < \alpha \leq 1$, *and* a, b *are real numbers, then*

$$E_\alpha(at^\alpha) \otimes cos_\alpha(bt^\alpha) = \frac{1}{2}[E_\alpha((a+ib)t^\alpha) + E_\alpha((a-ib)t^\alpha)], \tag{12}$$

and

$$E_\alpha(at^\alpha) \otimes sin_\alpha(bt^\alpha) = \frac{1}{2i}\left[E_\alpha\left((a+ib)t^\alpha\right) - E_\alpha\left((a-ib)t^\alpha\right)\right]. \tag{13}$$

Proof Since $E_\alpha((a+ib)t^\alpha)$

$= E_\alpha(at^\alpha) \otimes E_\alpha(ibt^\alpha)$ (by Eq. (11))

$= E_\alpha(at^\alpha) \otimes [cos_\alpha(bt^\alpha) + isin_\alpha(bt^\alpha)]$ (by Eq. (4))

$= E_\alpha(at^\alpha) \otimes cos_\alpha(bt^\alpha) + iE_\alpha(at^\alpha) \otimes isin_\alpha(bt^\alpha)$.

It follows that

$$E_\alpha(at^\alpha) \otimes cos_\alpha(bt^\alpha) = \mathrm{Re}[E_\alpha((a+ib)t^\alpha)] = \frac{1}{2}[E_\alpha((a+ib)t^\alpha) + E_\alpha((a-ib)t^\alpha)].$$

And

$$E_\alpha(at^\alpha) \otimes sin_\alpha(bt^\alpha) = \mathrm{Im}[E_\alpha((a+ib)t^\alpha)] = \frac{1}{2i}[E_\alpha((a+ib)t^\alpha) - E_\alpha((a-ib)t^\alpha)].$$

q.e.d.

Proposition 2.13: *If* $0 < \alpha \leq 1$, *b is a real number, and m is a positive integer, then*

$$cos_\alpha^{\otimes m}(bt^\alpha) = \frac{1}{2^m}\sum_{l=0}^{m} \frac{m!}{l!(m-l)!} cos_\alpha((m-2l)bt^\alpha), \tag{14}$$

and

$$sin_\alpha^{\otimes m}(bt^\alpha) = \frac{1}{(-2)^m}\sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!}\left(cos\frac{m\pi}{2} \cdot cos_\alpha((m-2l)bt^\alpha) - sin\frac{m\pi}{2} \cdot sin_\alpha((m-2l)bt^\alpha)\right). \tag{15}$$

Proof

$$cos_\alpha^{\otimes m}(bt^\alpha) = \left(\frac{1}{2}\left(E_\alpha(ibt^\alpha) + E_\alpha(-ibt^\alpha)\right)\right)^{\otimes m}$$

$$= \frac{1}{2^m}\sum_{l=0}^{m} \frac{m!}{l!(m-l)!}\left(E_\alpha(ibt^\alpha)\right)^{\otimes m-l} \otimes \left(E_\alpha(-ibt^\alpha)\right)^{\otimes l}$$

$$= \frac{1}{2^m}\sum_{l=0}^{m} \frac{m!}{l!(m-l)!} E_\alpha(i(m-l)bt^\alpha) \otimes E_\alpha(-ilbt^\alpha)$$

$$= \frac{1}{2^m}\sum_{l=0}^{m} \frac{m!}{l!(m-l)!} E_\alpha(i(m-2l)bt^\alpha)$$

$$= \frac{1}{2^m}\sum_{l=0}^{m} \frac{m!}{l!(m-l)!} cos_\alpha((m-2l)bt^\alpha). \quad \text{(by Eq. (4))}$$

And

$$sin_\alpha^{\otimes m}(bt^\alpha)$$

$$= \left(\frac{1}{2i}\left(E_\alpha(ibt^\alpha) - E_\alpha(-ibt^\alpha)\right)\right)^{\otimes m}$$

$$= \frac{1}{(2i)^m}\sum_{l=0}^{m} \frac{m!}{l!(m-l)!}\left(E_\alpha(ibt^\alpha)\right)^{\otimes m-l} \otimes \left(-E_\alpha(-ibt^\alpha)\right)^{\otimes l}$$

$$= \frac{i^m}{(-2)^m}\sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!} E_\alpha(i(m-2l)bt^\alpha)$$

$$= \frac{1}{(-2)^m}\sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!}\left(cos\frac{m\pi}{2} + isin\frac{m\pi}{2}\right)\left(cos_\alpha((m-2l)bt^\alpha) + isin_\alpha((m-2l)bt^\alpha)\right)$$

$$= \frac{1}{(-2)^m}\sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!}\left(cos\frac{m\pi}{2} \cdot cos_\alpha((m-2l)bt^\alpha) - sin\frac{m\pi}{2} \cdot sin_\alpha((m-2l)bt^\alpha)\right).$$

q.e.d.

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III. MAIN RESULTS

In this section, we will find the integral form of particular solution of non-homogeneous linear FDE with constant coefficients. Firstly, we introduce the fractional derivatives adopted in this paper.

Notation 3.1: If α is a real number, we define

$$[\alpha]=\begin{cases} 0 & \text{if } \alpha<0, \\ \text{the greatest integer less than or equal to } \alpha & \text{if } \alpha\geq 0. \end{cases}$$

Definition 3.2: Assume that α is a real number, m is a positive integer, and $f(t)\in C^{[\alpha]}([a,b])$. The modified Riemann-Liouville fractional derivatives of Jumarie type ([13, 15]) is defined by

$$\left({}_aD_t^{\alpha}\right)[f(t)]=\begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^t (t-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau, & \text{if } 0<\alpha<1 \\ \frac{d^m}{dt^m}\left({}_aD_t^{\alpha-m}\right)[f(t)], & \text{if } m\leq\alpha<m+1 \end{cases} \tag{16}$$

where $\Gamma(y)=\int_0^{\infty}s^{y-1}e^{-s}ds$ is the gamma function defined on $y>0$. For any positive integer n, we define $\left({}_aD_t^{\alpha}\right)^n=\left({}_aD_t^{\alpha}\right)\left({}_aD_t^{\alpha}\right)\cdots\left({}_aD_t^{\alpha}\right)$, the n-th order fractional derivative of ${}_aD_t^{\alpha}$. On the other hand, the fractional integral is defined by ${}_aI_t^{\alpha}={}_aD_t^{-\alpha}$, where $\alpha>0$. We have the following properties.

Proposition 3.3 ([19])**:** *Suppose that α,β,c are real constants and $0<\alpha\leq 1$, then*

$$\left({}_0D_t^{\alpha}\right)\left[t^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha},\ \text{if } \beta\geq\alpha \tag{17}$$

$$\left({}_0D_t^{\alpha}\right)[c]=0, \tag{18}$$

$$\left({}_0I_t^{\alpha}\right)\left[t^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha},\ \text{if } \beta>-1. \tag{19}$$

Theorem 3.4 ([19])**:** *If $0<\alpha\leq 1$ and $f(t)$ is a continuous function, then*

$$\left({}_aD_t^{\alpha}\right)\left({}_aI_t^{\alpha}\right)[f(t)]=f(t). \tag{20}$$

Theorem 3.5: *If a,b,α are real constants, $a^2+b^2\neq 0$, and $0<\alpha\leq 1$, then the fractional integrals*

$$\left({}_0I_t^{\alpha}\right)[E_{\alpha}(at^{\alpha})\otimes cos_{\alpha}(bt^{\alpha})]=\frac{1}{a^2+b^2}E_{\alpha}(at^{\alpha})\otimes\left(acos_{\alpha}(bt^{\alpha})+bsin_{\alpha}(bt^{\alpha})\right)-\frac{a}{a^2+b^2}. \tag{21}$$

And

$$\left({}_0I_t^{\alpha}\right)[E_{\alpha}(at^{\alpha})\otimes sin_{\alpha}(bt^{\alpha})]=\frac{-1}{a^2+b^2}E_{\alpha}(at^{\alpha})\otimes\left(bcos_{\alpha}(bt^{\alpha})-asin_{\alpha}(bt^{\alpha})\right)+\frac{b}{a^2+b^2}. \tag{22}$$

Proof $\left({}_0I_t^{\alpha}\right)[E_{\alpha}(at^{\alpha})\otimes cos_{\alpha}(bt^{\alpha})]$

$=\left({}_0I_t^{\alpha}\right)\left[\frac{1}{2}[E_{\alpha}((a+ib)t^{\alpha})+E_{\alpha}((a-ib)t^{\alpha})]\right]$ (by Eq. (11))

$=\frac{1}{2}\left({}_0I_t^{\alpha}\right)[E_{\alpha}((a+ib)t^{\alpha})+E_{\alpha}((a-ib)t^{\alpha})]$

$=\frac{1}{2}\left(\frac{1}{a+ib}E_{\alpha}((a+ib)t^{\alpha})+\frac{1}{a-ib}E_{\alpha}((a-ib)t^{\alpha})-\frac{1}{a+ib}-\frac{1}{a-ib}\right)$

$$=\frac{1}{2(a^2+b^2)}\left((a-ib)E_{\alpha}((a+ib)t^{\alpha})+(a+ib)E_{\alpha}((a-ib)t^{\alpha})\right)-\frac{a}{a^2+b^2}$$

$$=\frac{1}{2(a^2+b^2)}\left((a-ib)\left(E_{\alpha}(at^{\alpha})\otimes\left(cos_{\alpha}(bt^{\alpha})+isin_{\alpha}(bt^{\alpha})\right)\right)+(a+ib)\left(E_{\alpha}(at^{\alpha})\otimes\left(cos_{\alpha}(bt^{\alpha})-isin_{\alpha}(bt^{\alpha})\right)\right)\right)-\frac{a}{a^2+b^2}$$

$=\frac{1}{a^2+b^2}E_{\alpha}(at^{\alpha})\otimes\left(acos_{\alpha}(bt^{\alpha})+bsin_{\alpha}(bt^{\alpha})\right)-\frac{a}{a^2+b^2}$.

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On the other hand,

$$\begin{aligned}
&\left({}_0I_t^{\alpha}\right)[E_\alpha(at^\alpha)\otimes sin_\alpha(bt^\alpha)]\\
&=\left({}_0I_t^{\alpha}\right)\left[\tfrac{1}{2i}[E_\alpha((a+ib)t^\alpha)-E_\alpha((a-ib)t^\alpha)]\right] \text{ (by Eq. (12))}\\
&=\tfrac{1}{2i}\left(\tfrac{1}{a+ib}E_\alpha((a+ib)t^\alpha)-\tfrac{1}{a-ib}E_\alpha\left((a-ib)t^\alpha\right)-\tfrac{1}{a+ib}+\tfrac{1}{a-ib}\right)\\
&=\tfrac{-1}{2(a^2+b^2)}\left((b+ai)E_\alpha((a+ib)t^\alpha)+(b-ai)E_\alpha((a-ib)t^\alpha)\right)+\tfrac{b}{a^2+b^2}\\
&=\tfrac{-1}{a^2+b^2}E_\alpha(at^\alpha)\otimes\left(bcos_\alpha(bt^\alpha)-asin_\alpha(bt^\alpha)\right)+\tfrac{b}{a^2+b^2}.
\end{aligned}$$

q.e.d.

The linearity property obviously holds by Definition 3.2.

$$\left({}_0D_t^{\alpha}\right)[af_\alpha(\lambda t^\alpha)+bg_\alpha(\mu t^\alpha)]=a\left({}_0D_t^{\alpha}\right)[f_\alpha(\lambda t^\alpha)]+b\left({}_0D_t^{\alpha}\right)[g_\alpha(\mu t^\alpha)], \tag{23}$$

where f_α, g_α are fractional functions, and a, b, λ, μ are complex constants.

Theorem 3.6 (product rule for fractional derivatives) ([17])**:** *Let* $0 < \alpha \le 1$, λ, μ *be complex numbers, and* f_α, g_α *be fractional functions, then*

$$\left({}_0D_t^{\alpha}\right)[f_\alpha(\lambda t^\alpha)\otimes g_\alpha(\mu t^\alpha)]=\left({}_0D_t^{\alpha}\right)[f_\alpha(\lambda t^\alpha)]\otimes g_\alpha(\mu t^\alpha)+f_\alpha(\lambda t^\alpha)\otimes\left({}_0D_t^{\alpha}\right)[g_\alpha(\mu t^\alpha)]. \tag{24}$$

The following is the differential form of particular solution of non-homogeneous linear FDE with constant coefficients.

Theorem 3.7 ([18])**:** *If* $0 < \alpha \le 1, n$ *is a positive integer,* $a_0.a_1,\cdots,a_n$ *are real constants, and* $a_n \ne 0$. *The non-homogeneous linear FDE with constant coefficients*

$$\left(a_n\left({}_0D_t^{\alpha}\right)^n+a_{n-1}\left({}_0D_t^{\alpha}\right)^{n-1}+\cdots+a_1\left({}_0D_t^{\alpha}\right)+a_0\right)[y(t^\alpha)]=r(t^\alpha) \tag{25}$$

has the particular solution

$$y_p(t^\alpha)=\left(\frac{1}{a_n\left({}_0D_t^{\alpha}\right)^n+a_{n-1}\left({}_0D_t^{\alpha}\right)^{n-1}+\cdots+a_1\left({}_0D_t^{\alpha}\right)+a_0}\right)[r(t^\alpha)]. \tag{26}$$

To obtain the main result of this article, we need the following properties.

Lemma 3.8: *Let* λ *be a complex number, then*

$$\left(\frac{1}{{}_0D_t^{\alpha}-\lambda}\right)[r(t^\alpha)]=E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]. \tag{27}$$

Proof

$$\begin{aligned}
&\left({}_0D_t^{\alpha}-\lambda\right)\left[E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]\right]\\
&=\left({}_0D_t^{\alpha}\right)\left[E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]\right]-\lambda E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]\\
&=\left({}_0D_t^{\alpha}\right)[E_\alpha(\lambda t^\alpha)]\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]+E_\alpha(\lambda t^\alpha)\otimes\left(E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)\right)\\
&\quad-\lambda E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)] \text{ (by Theorem 3.6)}\\
&=\lambda E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]+r(t^\alpha)-\lambda E_\alpha(\lambda t^\alpha)\otimes\left({}_0I_t^{\alpha}\right)[E_\alpha(-\lambda t^\alpha)\otimes r(t^\alpha)]\\
&=r(t^\alpha).
\end{aligned}$$

By Theorem 3.7, the desired result holds.

q.e.d.

Theorem 3.9: *Let the assumptions be the same as Lemma 3.8, and* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be the roots of the equation* $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$, *where* $a_n \ne 0$. *Then*

$$\left(\frac{1}{a_n\left({}_0D_t^{\alpha}\right)^n+a_{n-1}\left({}_0D_t^{\alpha}\right)^{n-1}+\cdots+a_1\left({}_0D_t^{\alpha}\right)+a_0}\right)[r(t^\alpha)]$$

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$$= \frac{1}{a_n} \cdot E_\alpha(\lambda_n t^\alpha) \otimes ({}_0I_t^\alpha) [E_\alpha((\lambda_{n-1} - \lambda_n) t^\alpha) \otimes \cdots E_\alpha((\lambda_1 - \lambda_2) t^\alpha) \otimes ({}_0I_t^\alpha) ({}_0I_t^\alpha) [E_\alpha(-\lambda_1 t^\alpha) \otimes r(t^\alpha)] \cdots]. \tag{28}$$

Proof $\left(\frac{1}{a_n\left({}_0D_x^\alpha\right)^n + a_{n-1}\left({}_0D_x^\alpha\right)^{n-1} + \cdots + a_1\left({}_0D_x^\alpha\right) + a_0}\right)[r(t^\alpha)]$

$$= \left(\frac{1}{a_n\left({}_0D_t^\alpha - \lambda_n\right)\left({}_0D_t^\alpha - \lambda_{n-1}\right)\cdots\left({}_0D_t^\alpha - \lambda_1\right)}\right)[r(t^\alpha)]$$

$$= \frac{1}{a_n} \cdot \left(\frac{1}{{}_0D_t^\alpha - \lambda_n}\right)\left(\frac{1}{{}_0D_t^\alpha - \lambda_{n-1}}\right)\cdots\left(\frac{1}{{}_0D_t^\alpha - \lambda_1}\right)[r(t^\alpha)]$$

$$= \frac{1}{a_n} \cdot \left(\frac{1}{{}_0D_t^\alpha - \lambda_n}\right)\left(\frac{1}{{}_0D_t^\alpha - \lambda_{n-1}}\right)\cdots\left(\frac{1}{{}_0D_t^\alpha - \lambda_2}\right)\left[E_\alpha(\lambda_1 t^\alpha) \otimes ({}_0I_t^\alpha) [E_\alpha(-\lambda_1 t^\alpha) \otimes r(t^\alpha)]\right] \quad \text{(by Lemma 3.8)}$$

$$= \frac{1}{a_n} \cdot \left(\frac{1}{{}_0D_t^\alpha - \lambda_n}\right)\left(\frac{1}{{}_0D_t^\alpha - \lambda_{n-1}}\right)\cdots\left(\frac{1}{{}_0D_t^\alpha - \lambda_3}\right)\left[E_\alpha(\lambda_2 t^\alpha) \otimes ({}_0I_t^\alpha) \left[E_\alpha((\lambda_1 - \lambda_2) t^\alpha) \otimes ({}_0I_t^\alpha) [E_\alpha(-\lambda_1 t^\alpha) \otimes r(t^\alpha)]\right]\right]$$

$$= \frac{1}{a_n} \cdot E_\alpha(\lambda_n t^\alpha) \otimes ({}_0I_t^\alpha) [E_\alpha((\lambda_{n-1} - \lambda_n) t^\alpha) \otimes \cdots E_\alpha((\lambda_1 - \lambda_2) t^\alpha) \otimes ({}_0I_t^\alpha) ({}_0I_t^\alpha) [E_\alpha(-\lambda_1 t^\alpha) \otimes r(t^\alpha)] \cdots].$$

q.e.d.

Therefore, we can easily obtain the integral form of particular solution of non-homogeneous linear FDE with constant coefficients.

Theorem 3.10: *The non-homogeneous linear FDE with constant coefficients*

$$\left(a_n\left({}_0D_t^\alpha\right)^n + a_{n-1}\left({}_0D_t^\alpha\right)^{n-1} + \cdots + a_1\left({}_0D_t^\alpha\right) + a_0\right)[y(t^\alpha)] = r(t^\alpha) \tag{29}$$

has the particular solution

$$y_p(t^\alpha) = \frac{1}{a_n} \cdot E_\alpha(\lambda_n t^\alpha) \otimes ({}_0I_t^\alpha) \left[E_\alpha((\lambda_{n-1} - \lambda_n) t^\alpha) \otimes ({}_0I_t^\alpha) [\cdots E_\alpha((\lambda_1 - \lambda_2) t^\alpha) \otimes ({}_0I_t^\alpha) [E_\alpha(-\lambda_1 t^\alpha) \otimes r(t^\alpha)] \cdots]\right]. \tag{30}$$

Proof By Theorem 3.7 and Theorem 3.9, the desired result holds. q.e.d.

Remark 3.11: The particular solution Eq. (30) may contain the general solution part of

$$\left(a_n\left({}_0D_t^\alpha\right)^n + a_{n-1}\left({}_0D_t^\alpha\right)^{n-1} + \cdots + a_1\left({}_0D_t^\alpha\right) + a_0\right)[y(t^\alpha)] = 0. \tag{31}$$

So we can ignore the general solution part when we seek the particular solution Eq. (30), and the particular solution that the general solution part to be removed denoted as $P\left(y_p(t^\alpha)\right)$.

IV. EXAMPLES

In the following, we will give several examples to illustrate the integral form of particular solution of non-homogeneous linear FDE with constant coefficients.

Example 4.1: Let a, b be real numbers, and $a \neq b$. The non-homogeneous linear FDE with constant coefficients

$$\left(\left({}_0D_t^{1/3}\right)^2 - 2a\left({}_0D_t^{1/3}\right) + a^2\right)\left[y\left(t^{1/3}\right)\right] = E_{1/3}\left(bt^{1/3}\right) \tag{32}$$

has the particular solution

$$P\left(y_p\left(t^{1/3}\right)\right)$$

$$= P\left(E_{1/3}\left(at^{1/3}\right) \otimes \left({}_0I_t^{1/3}\right)\left[\left({}_0I_t^{1/3}\right)\left[E_{1/3}\left((b-a)t^{1/3}\right)\right]\right]\right)$$

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$$= P\left(E_{1/3}\left(at^{1/3}\right) \otimes \left({}_0I_t^{1/3}\right)\left[\frac{1}{b-a} E_{1/3}\left((b-a)t^{1/3}\right)\right]\right)$$

$$= P\left(E_{1/3}\left(at^{1/3}\right) \otimes \left(\tfrac{1}{(b-a)^2} E_{1/3}\left((b-a)t^{1/3}\right)\right)\right)$$

$$= \tfrac{1}{(b-a)^2} E_{1/3}\left(bt^{1/3}\right). \tag{33}$$

Example 4.2: Let $t \geq 0$, then

$$\left(\left({}_0D_t^{1/4}\right)^2 - 3\left({}_0D_t^{1/4}\right) - 4\right)\left[y\left(t^{1/4}\right)\right] = E_{1/4}\left(-5t^{1/4}\right) \otimes sin_{1/4}\left(2t^{1/4}\right) \tag{34}$$

has the particular solution

$$P\left(y_p\left(t^{1/4}\right)\right)$$

$$= P\left(E_{1/4}\left(4t^{1/4}\right) \otimes \left({}_0I_t^{1/4}\right)\left[E_{1/4}\left(-5t^{1/4}\right) \otimes \left({}_0I_t^{1/4}\right)\left[E_{1/4}\left(-4t^{1/4}\right) \otimes sin_{1/4}\left(2t^{1/4}\right)\right]\right]\right)$$

$$= P\left(E_{1/4}\left(4t^{1/4}\right) \otimes \left({}_0I_t^{1/4}\right)\left[\tfrac{-1}{20} E_{1/4}\left(-9t^{1/4}\right) \otimes \left(2cos_{1/4}\left(2t^{1/4}\right) + 4sin_{1/4}\left(2t^{1/4}\right)\right)\right]\right) \quad \text{(by Theorem 3.5)}$$

$$= P\left(\tfrac{-1}{20} E_{1/4}\left(4t^{1/4}\right) \otimes \left({}_0I_t^{1/4}\right)\left[E_{1/4}\left(-9t^{1/4}\right) \otimes \left(2cos_{1/4}\left(2t^{1/4}\right) + 4sin_{1/4}\left(2t^{1/4}\right)\right)\right]\right) \quad \text{(by Theorem 3.5)}$$

$$= E_{1/4}\left(-5t^{1/4}\right) \otimes \left(\tfrac{26}{1700} cos_{1/4}\left(2t^{1/4}\right) + \tfrac{32}{1700} sin_{1/4}\left(2t^{1/4}\right)\right). \tag{35}$$

Example 4.3: $\left({}_0D_t^{1/3}\right)^2\left[y\left(t^{1/3}\right)\right] = cos_{1/3}^{\otimes 3}\left(2t^{1/3}\right)$ (36)

has the particular solution

$$P\left(y_p\left(t^{1/3}\right)\right)$$

$$= P\left(\left({}_0I_t^{1/3}\right)^2\left[cos_{1/3}^{\otimes 3}\left(2t^{1/3}\right)\right]\right)$$

$$= P\left(\left({}_0I_t^{1/3}\right)^2\left[\tfrac{1}{4} cos_{1/3}\left(6t^{1/3}\right) + \tfrac{3}{4} cos_{1/3}\left(2t^{1/3}\right)\right]\right) \quad \text{(by Eq. (14))}$$

$$= P\left(\left({}_0I_t^{1/3}\right)\left[\tfrac{1}{24} sin_{1/3}\left(6t^{1/3}\right) + \tfrac{3}{8} sin_{1/3}\left(2t^{1/3}\right)\right]\right)$$

$$= \tfrac{-1}{144} cos_{1/3}\left(6t^{1/3}\right) - \tfrac{3}{16} cos_{1/3}\left(2t^{1/3}\right). \tag{37}$$

Example 4.4: $\left({}_0D_t^{1/5} + 1\right)\left[y\left(t^{1/5}\right)\right] = t^{1/5}$

(38)

has the particular solution

$$P\left(y_p\left(t^{1/5}\right)\right) = P\left(E_{1/5}\left(-t^{1/5}\right) \otimes \left({}_0I_t^{1/5}\right)\left[E_{1/5}\left(t^{1/5}\right) \otimes t^{1/5}\right]\right)$$

$$= P\left(E_{1/5}\left(-t^{1/5}\right) \otimes \left(E_{1/5}\left(t^{1/5}\right) \otimes \left(t^{1/5} - \Gamma\left(\tfrac{6}{5}\right)\right)\right)\right)$$

$$= t^{1/5} - \Gamma\left(\tfrac{6}{5}\right). \tag{39}$$

V. CONCLUSIONS

The particular solution of non-homogeneous linear FDE with constant coefficients has two expressions: differential form and integral form. They are generalizations of particular solution of non-homogeneous linear ordinary differential equation with constant coefficients. In this paper, the integral form is proved by product rule of fractional functions. On the other hand, the differential form can be represented as formal Laurent series of fractional differential operator. In fact, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of engineering mathematics.

REFERENCES

[1] Diethelm, K. , Freed, A.D., "On the Solution of Nonlinear Fractional Order Differential Equations Used in the Modeling of Viscoplasticity", Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer, pp. 217-224, 1999.

[2] Sheng, H., Chen, Y., Qiu, T., " Fractional Processes and Fractional-Order Signal Processing", Techniques and Applications, Springer-Verlag, London, 2011.

[3] Vong, S. W., " Positive Solutions of Singular Fractional Differential Equations with Integral Boundary Conditions", Mathematical and Computer Modelling, Vol. 57, pp. 1053-1059, 2013.

[4] Samko, S. G., Kilbas, A. A, Marichev, O. I. " Fractional Integrals and Derivatives: Theory and Applications", Amsterdam, The Netherlands: Gordon and Breach, 1993.

[5] Das, S., " Functional Fractional Calculus", 2nd Edition, Springer-Verlag, 2011.

[6] Metzler, F., Schick, W., Kilian, H.G., Nonnenmacher, T.F., "Relaxation in Filled Polymers: A Fractional Calculus Approach", Journal of Chemical Physics, Vol. 103, pp. 7180-7186, 1995.

[7] Gaul, L., Klein, P., Kempfle, S., "Damping Description Involving Fractional Operators", Mechanical Systems and Signal Processing, Vol. 5, pp. 81-88, 1991.

[8] Duarte, F., Machado, J. A. T., "Chaotic Phenomena and Fractional-Order Dynamics in the Trajectory Control of Redundant Manipulators", Nonlinear Dynamics, Vol. 29, No. 1-4, pp. 315-342, 2002.

[9] Machado, J. A. T., "Analysis and Design of Fractional-Order Digital Control Systems", Systems Analysis Modelling Simulation, Vol. 27, No. 2-3, pp. 107-122, 1997.

[10] Glockle, W.G., Nonnenmacher, T.F., "A Fractional Calculus Approach of Self-Similar Protein Dynamics", Biophysical Journal, Vol. 68, pp.46-53, 1995.

[11] Jiang, Y., Ding, X., "Waveform Relaxation Methods for Fractional Differential Equations with the Caputo Derivatives", Journal of Computational and Applied Mathematics, Vol. 238, pp. 51-67, 2013.

[12] Podlubny, I., "Fractional Differential Equations", Academic Press, San Diego, 1999.

[13] Jumarie, G., "Modified Riemann-Liouville Derivative and Fractional Taylor Series of Nondifferentiable Functions Further Results", Computers & Mathematics with Applications, Vol. 51, Issue 9, pp.1367-76, 2006.

[14] Prajapati, J. C., "Certain Properties of Mittag-Leffler Function with Argument x^α, α > 0", Italian Journal of Pure and Applied Mathematics, No. 30, pp. 411-416, 2013.

[15] Ghosh, U., Sengupta, S., Sarkar, S., Das, S., " Analytic Solution of Linear Fractional Differential Equation with Jumarie Derivative in Term of Mittag-Leffler Function", American Journal of Mathematical Analysis, Vol. 3, No. 2, pp. 32-38, 2015.

[16] Khalaf, S. L., "Particular Solution of Linear Fractional Differential Equation with Constant Coefficients by Inverse Operators", International Journal of Pure and Applied Research in Engineering and Technology, Vol. 4, Issue 7, pp. 48-63, 2016.

[17] Yu, C. -H., "Differential Properties of Fractional Functions", Communications in Nonlinear Science and Numerical Simulation, prepare.

[18] Yu, C. -H., "Fractional Derivatives of Some Fractional Functions and Their Applications", Asian Journal of Applied Science and Technology, Vol. 4, Issue 1, pp. 147-158, 2020.

[19] Syam, M. I., Alquran, M., Jaradat, H. M., Al-Shara, S., "The Modified Fractional Power Series for Solving a Class of Fractional Sturm-Liouville Eigenvalue Problems", Journal of Fractional Calculus and Applications, Vol. 10, Issue 1, pp. 154-166, 2019.

[20] Peng, J, Li, K,"A Note on Property of the Mittag-Leffler Function", Journal of Mathematical Analysis and Applications, Vol. 370, Issue 2, pp. 635-638, 2010.

[21] Area, I, Losada, Juan J., Nieto, J. J.," A Note on the Fractional Logistic Equation", Physica A: Statistical Mechanics and its Applications, Vol. 444, pp. 182-187, 2016.K.B. Oldham and J. Spanier, The Fractional Calculus, Academic, New York, 1974.