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Differential Properties of Fractional Functions

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***Abstract*: In this paper, we solve the important problem that some fractional differential properties do not hold for general functions. A new multiplication is introduced such that the fractional functions satisfy these basic properties, regarding the modified Riemann-Liouville (R-L) fractional derivatives. Therefore, we can obtain the fractional derivatives of some fractional functions, including fractional trigonometric functions and hyperbolic functions. Furthermore, several examples are proposed for demonstrating the advantage of our method.**

***Keywords*: Fractional differential properties, Modified R-L fractional derivative, Fractional functions.**

I. INTRODUCTION

The fractional calculus started in 1695, Leibniz wrote a letter to L'Hôpital raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order derivatives such as $\frac{d^{1/2}}{dx^{1/2}}y$. The eminent mathematicians such as Fourier, Abel, Liouville, Riemann, Weyl, Riesz, and many others contributed to this field. For a long time, fractional calculus has been regarded as a pure mathematical realm without real applications. But, in recent decades, such a state of affairs has been changed. The fractional calculus have been applied in widespread fields of science and engineering. It has been found that fractional calculus can be useful and even powerful, and an outline of the simple history about fractional calculus can be found in [1-7]. In fact, many scientific areas are currently paying attention to the fractional calculus concepts and we can refer its adoption in viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modeling and identification, telecommunications, chemistry, irreversibility, physics, and control systems [8-14]. Unlike standard calculus, there is no unique definition of derivation and integration in fractional calculus. The commonly used definition is the Riemann-Liouville (R-L) fractional derivative [4]. Other useful definitions include Caputo definition of fractional derivative (1967) [7], the Grunwald-Letinikov (G-L) fractional derivative [4], and Jumarie's modified R-L fractional derivative is used to avoid nonzero fractional derivative of constant functions [15].

Liu [16] gave two examples to show that the product rule and chain rule of Jumarie's modified R-L fractional derivative are incorrect for general functions. The main aim of this paper is to provide a new multiplication and show that under this new multiplication, the fractional functions obey classical differential properties such as linearity, product rule, quotient rule, chain rule, Leibniz rule, and Faà di Bruno formula. Therefore, we can find the fractional differential properties mentioned above for several fractional functions, for example, fractional trigonometric functions and hyperbolic functions. In addition, some examples are given to demonstrate the validity of our results.

On the other hand, the differential equations in different form of fractional derivatives give different type of solutions. So, there is no standard methods to solve fractional differential equations (FDE). Ghosh et al. [17] developed analytical method for solution of linear fractional differential equations with Jumarie type of modified R-L derivative. This manuscript is dedicated to use the formal Laurent series fractional differential operator method to find the particular solution of non-homogeneous linear FDE with constant coefficients. This method is different from [18-19] and a generalization of the approach proposed by [20]. The Mittag-Leffler function plays an important role in this paper, and the results obtained by us are the generalizations of traditional differential calculus cases.

II. PRELIMINARIES

At first, we introduce some fractional functions.

Definition 2.1 ([22])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{1}$$

where α is a real number, $\alpha > 0$, and z is a complex variable.

Definition 2.2 ([17])**:** $E_\alpha(\lambda x^\alpha)$ is called α-order fractional exponential function. The α-order fractional cosine and sine function are defined as follows:

$$cos_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{2}$$

and

$$sin_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{3}$$

where $0 < \alpha \leq 1$, λ is a complex number, and x is a real variable.

The following is a new multiplication of fractional functions.

Definition 2.3: Let λ, μ, z be complex numbers, $0 < \alpha \leq 1$, j, l, k be non-negative integers, and a_k, b_k be real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)} z^k$ for all k. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha) = \frac{1}{\Gamma(j\alpha+1)} (\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)} (\mu y^\alpha)^l = \frac{1}{\Gamma((j+l)\alpha+1)} \binom{j+l}{j} (\lambda x^\alpha)^j (\mu y^\alpha)^l, \tag{4}$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

If $f_\alpha(\lambda x^\alpha)$ and $g_\alpha(\mu y^\alpha)$ are two fractional functions,

$$f_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\lambda x^\alpha)^k, \tag{5}$$

$$g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\mu y^\alpha)^k, \tag{6}$$

then we define

$$f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu y^\alpha) \right). \tag{7}$$

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Proposition 2.4: $$f_\alpha(\lambda x^\alpha)\otimes g_\alpha(\mu y^\alpha)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\,(\lambda x^\alpha)^{k-m}(\mu y^\alpha)^m. \quad (8)$$

Proof $$f_\alpha(\lambda x^\alpha)\otimes g_\alpha(\mu y^\alpha)$$

$$=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}a_{k-m}b_m p_{k-m}(\lambda x^\alpha)\otimes p_m(\mu y^\alpha)\right)$$

$$=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}a_{k-m}b_m\frac{1}{\Gamma((k-m)\alpha+1)}(\lambda x^\alpha)^{k-m}\otimes\frac{1}{\Gamma(m\alpha+1)}(\mu y^\alpha)^m\right)$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\,(\lambda x^\alpha)^{k-m}(\mu y^\alpha)^m\,.$$ q.e.d.

Definition 2.5: Let $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes n}=f_\alpha(\lambda x^\alpha)\otimes\cdots\otimes f_\alpha(\lambda x^\alpha)$ be the n times product of the fractional function $f_\alpha(\lambda x^\alpha)$. If $f_\alpha(\lambda x^\alpha)\otimes g_\alpha(\lambda x^\alpha)=1$, then $g_\alpha(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^\alpha)$, and is denoted by $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes -1}$.

Remark 2.6: The $\otimes$ multiplication satisfies the commutative law and the associate law, and is the generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha=1$.

Proposition 2.7: $$E_\alpha(\lambda x^\alpha)\otimes E_\alpha(\mu y^\alpha)=E_\alpha(\lambda x^\alpha+\mu y^\alpha). \quad (9)$$

Proof $$E_\alpha(\lambda x^\alpha)\otimes E_\alpha(\mu y^\alpha)$$

$$=\sum_{k=0}^{\infty}\frac{(\lambda x^\alpha)^k}{\Gamma(k\alpha+1)}\otimes\sum_{k=0}^{\infty}\frac{(\mu y^\alpha)^k}{\Gamma(k\alpha+1)}$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m}(\lambda x^\alpha)^{k-m}(\mu y^\alpha)^m$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}(\lambda x^\alpha+\mu y^\alpha)^k$$

$$=E_\alpha(\lambda x^\alpha+\mu y^\alpha)\,.$$ q.e.d.

Corollary 2.8: $$E_\alpha(\lambda x^\alpha)\otimes E_\alpha(\mu x^\alpha)=E_\alpha((\lambda+\mu)x^\alpha). \quad (10)$$

Remark 2.9: Peng and Li [24] give an example to show that $E_\alpha(\lambda x^\alpha)\cdot E_\alpha(\lambda x^\alpha)=E_\alpha(\lambda(x+y)^\alpha)$ is not true for $0<\alpha<1$. On the other hand, Area, *et al*. [25] also provide a counterexample for $E_\alpha(\lambda x^\alpha)\cdot E_\alpha(\mu x^\alpha)=E_\alpha((\lambda+\mu)x^\alpha)$, $0<\alpha<1$.

Proposition 2.10 (fractional Euler's formula) ([23])**:** *Let* $0<\alpha\le 1$, *then*

$$E_\alpha(ix^\alpha)=cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)\,. \quad (11)$$

Proposition 2.11 (fractional DeMoivre's formula): *If* $0<\alpha\le 1$, *and* n *is a positive integer, then*

$$[cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)]^{\otimes n}=cos_\alpha(nx^\alpha)+isin_\alpha(nx^\alpha)\,. \quad (12)$$

Proof $$[cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)]^{\otimes n}$$

International Journal of Novel Research in Interdisciplinary Studies
Vol. 7, Issue 5, pp: (1-14), Month: September - October 2020, Available at: www.noveltyjournals.com

$$= \left(E_\alpha(ix^\alpha)\right)^{\otimes n}$$

$$= E_\alpha(inx^\alpha)$$

$$= cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha).$$ q.e.d.

Next, we define the other fractional trigonometric functions.

Definition 2.12: Suppose that $0 < \alpha \leq 1$, and λ is a complex number, then

$$sec_\alpha^{\otimes}(\lambda x^\alpha) = \left(cos_\alpha(\lambda x^\alpha)\right)^{\otimes -1} \tag{13}$$

is called the α-order fractional secant function.

$$csc_\alpha^{\otimes}(\lambda x^\alpha) = \left(sin_\alpha(\lambda x^\alpha)\right)^{\otimes -1} \tag{14}$$

is the α-order fractional cosecant function.

$$tan_\alpha^{\otimes}(\lambda x^\alpha) = sin_\alpha(\lambda x^\alpha) \otimes sec_\alpha^{\otimes}(\lambda x^\alpha) \tag{15}$$

is called the α-order fractional tangent function.

$$cot_\alpha^{\otimes}(\lambda x^\alpha) = cos_\alpha(\lambda x^\alpha) \otimes csc_\alpha^{\otimes}(\lambda x^\alpha) \tag{16}$$

is the α-order fractional cotangent function.

The followings are the fractional hyperbolic functions.

Definition 2.13: Let $0 < \alpha \leq 1$, and λ be a complex number, then

$$cosh_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^{2k} x^{2k\alpha}}{\Gamma(2k\alpha+1)} \tag{17}$$

is the α-order fractional hyperbolic cosine function.

$$sinh_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^{2k+1} x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} \tag{18}$$

is the α-order fractional hyperbolic sine function.

$$sech_\alpha^{\otimes}(\lambda x^\alpha) = \left(cosh_\alpha(\lambda x^\alpha)\right)^{\otimes -1} \tag{19}$$

is the α-order fractional hyperbolic secant function.

$$csch_\alpha^{\otimes}(\lambda x^\alpha) = \left(sinh_\alpha(\lambda x^\alpha)\right)^{\otimes -1} \tag{20}$$

is the α-order fractional hyperbolic cosecant function.

$$tanh_\alpha^{\otimes}(\lambda x^\alpha) = sinh_\alpha(\lambda x^\alpha) \otimes sech_\alpha^{\otimes}(\lambda x^\alpha) \tag{21}$$

is the α-order fractional hyperbolic tangent function.

$$coth_\alpha^{\otimes}(\lambda x^\alpha) = cosh_\alpha(\lambda x^\alpha) \otimes csch_\alpha^{\otimes}(\lambda x^\alpha) \tag{22}$$

is the α-order fractional hyperbolic cotangent function.

III. METHODS AND RESULTS

In this section, the main methods and results are present. Firstly, we introduce the fractional derivatives adopted in this article.

3.1 Some fractional differential properties

Notation 3.1.1: If α is a real number, we define

$$[\alpha] = \begin{cases} 0 & \text{if } \alpha < 0, \\ \text{the greatest integer less than or equal to } \alpha & \text{if } \alpha \geq 0. \end{cases}$$

Definition 3.1.2: Suppose that α is a real number, m is a positive integer, and $f(x) \in C^{[\alpha]}([a, b])$. The modified Riemann-Liouville fractional derivatives of Jumarie type ([15, 21]) is defined by

$$\left({}_aD_x^\alpha\right)[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-\tau)^{-\alpha}[f(\tau) - f(a)]\, d\tau, & \text{if } 0 < \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m \leq \alpha < m+1, \end{cases} \tag{23}$$

where $\Gamma(y) = \int_0^\infty t^{y-1}e^{-t}dt$ is the gamma function defined on $y > 0$. And for any positive integer n, $\left({}_aD_x^\alpha\right)^n = \left({}_aD_x^\alpha\right)\left({}_aD_x^\alpha\right)\cdots\left({}_aD_x^\alpha\right)$ is the n-th order fractional derivative of ${}_aD_x^\alpha$. If $n = 0$, we define $\left({}_aD_x^\alpha\right)^0[f] = f$. We note that $\left({}_aD_x^\alpha\right)^n \neq {}_aD_x^{n\alpha}$ in general, and the following property holds.

Proposition 3.1.3 ([17])**:** *Assume that* α, β, c *are real constants and* $\beta \geq \alpha > 0$, *then*

$$\left({}_0D_x^\alpha\right)[x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{24}$$

and

$$\left({}_0D_x^\alpha\right)[c] = 0. \tag{25}$$

As mentioned above, the product rule and chain rule do not hold for traditional multiplication of general functions, regarding Jumarie's modified R-L fractional derivative ([16]). In the following, we show that the $\otimes$ multiplication satisfies linearity, product rule, quotient rule, Leibniz rule, chain rule, and Faà di Bruno formula. The following linearity property obviously holds by Definition 3.1.2.

$$\left({}_0D_x^\alpha\right)[af_\alpha(\lambda x^\alpha) + bg_\alpha(\mu x^\alpha)] = a\left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha)] + b\left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)], \tag{26}$$

where f_α , g_α are fractional functions, and a, b, λ, μ are constants.

To obtain the product rule for fractional derivatives, we need a lemma.

Lemma 3.1.4: *Suppose that* $0 < \alpha \leq 1$, λ, μ *are complex numbers,* j, l *are non-negative integers, then*

$$\left({}_0D_x^\alpha\right)[p_j(\lambda x^\alpha) \otimes p_l(\mu x^\alpha)] = \left({}_0D_x^\alpha\right)[p_j(\lambda x^\alpha)] \otimes p_l(\mu x^\alpha) + p_j(\lambda x^\alpha) \otimes \left({}_0D_x^\alpha\right)[p_l(\mu x^\alpha)]. \quad (27)$$

Proof

$$\left({}_0D_x^\alpha\right)[p_j(\lambda x^\alpha) \otimes p_l(\mu x^\alpha)]$$

$$= \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(j\alpha+1)}(\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu x^\alpha)^l\right]$$

$$= \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}\lambda^j\mu^l x^{(j+l)\alpha}\right]$$

$$= \frac{1}{\Gamma((j+l-1)\alpha+1)}\binom{j+l}{j}\lambda^j\mu^l x^{(j+l-1)\alpha}.$$

And,

$$\left({}_0D_x^\alpha\right)[p_j(\lambda x^\alpha)] \otimes p_l(\mu x^\alpha) + p_j(\lambda x^\alpha) \otimes \left({}_0D_x^\alpha\right)[p_l(\mu x^\alpha)]$$

$$= \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(j\alpha+1)}\lambda^j x^{j\alpha}\right] \otimes \frac{1}{\Gamma(l\alpha+1)}\mu^l x^{l\alpha} + \frac{1}{\Gamma(j\alpha+1)}\lambda^j x^{j\alpha} \otimes \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(l\alpha+1)}\mu^l x^{l\alpha}\right]$$

$$= \frac{1}{\Gamma((j-1)\alpha+1)}\lambda^j x^{(j-1)\alpha} \otimes \frac{1}{\Gamma(l\alpha+1)}\mu^l x^{l\alpha} + \frac{1}{\Gamma(j\alpha+1)}\lambda^j x^{j\alpha} \otimes \frac{1}{\Gamma((l-1)\alpha+1)}\mu^l x^{(l-1)\alpha}$$

$$= \frac{1}{\Gamma((j+l-1)\alpha+1)}\binom{j+l-1}{j-1}\lambda^j\mu^l x^{(j+l-1)\alpha} + \frac{1}{\Gamma((j+l-1)\alpha+1)}\binom{j+l-1}{j}\lambda^j\mu^l x^{(j+l-1)\alpha}$$

$$= \frac{1}{\Gamma((j+l-1)\alpha+1)}\binom{j+l}{j}\lambda^j\mu^l x^{(j+l-1)\alpha}.$$

Therefore, the desired result holds. q.e.d.

Theorem 3.1.5 (product rule for fractional derivatives): *Let* $0 < \alpha \leq 1$, λ, μ *be complex numbers, and* f_α , g_α *be fractional function. Then*

$$\left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu x^\alpha)] = \left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha)] \otimes g_\alpha(\mu x^\alpha) + f_\alpha(\lambda x^\alpha) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]. \quad (28)$$

Proof Let $f_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha)$ and $g_\alpha(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu x^\alpha)$, we have

$$\left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu x^\alpha)]$$

$$= \left({}_0D_x^\alpha\right)\left[\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m}b_m p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu x^\alpha)\right)\right]$$

$$= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m}b_m\left({}_0D_x^\alpha\right)[p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu x^\alpha)]\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m \left(\left({}_0D_x^{\alpha}\right)[p_{k-m}(\lambda x^{\alpha})] \otimes p_m(\mu x^{\alpha}) + p_{k-m}(\lambda x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[p_m(\mu x^{\alpha})]\right)\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m \left(\left({}_0D_x^{\alpha}\right)[p_{k-m}(\lambda x^{\alpha})] \otimes p_m(\mu x^{\alpha})\right)\right)$$

$$+ \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m \left(p_{k-m}(\lambda x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[p_m(\mu x^{\alpha})]\right)\right)$$

$$= \left({}_0D_x^{\alpha}\right)[f_{\alpha}(\lambda x^{\alpha})] \otimes g_{\alpha}(\mu x^{\alpha}) + f_{\alpha}(\lambda x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[g_{\alpha}(\mu x^{\alpha})].$$

And hence, the desired result holds. q.e.d.

Corollary 3.1.6: *Let $f_{\alpha,1},\cdots,f_{\alpha,p}$ be fractional functions and $\lambda_1,\lambda_2,\cdots,\lambda_p$ be complex numbers, then*

$$\left({}_0D_x^{\alpha}\right)\left[f_{\alpha,1}(\lambda_1 x^{\alpha}) \otimes \cdots \otimes f_{\alpha,p}(\lambda_p x^{\alpha})\right]$$

$$= \left({}_0D_x^{\alpha}\right)\left[f_{\alpha,1}(\lambda_1 x^{\alpha})\right] \otimes f_{\alpha,2} \otimes \cdots \otimes f_{\alpha,p} + \cdots + f_{\alpha,1} \otimes \cdots \otimes f_{\alpha,p-1} \otimes \left({}_0D_x^{\alpha}\right)\left[f_{\alpha,p}(\lambda_p x^{\alpha})\right]. \tag{29}$$

Corollary 3.1.7: *Suppose that f_{α} is a fractional function, m is a positive integer and λ is a complex number, then*

$$\left({}_0D_x^{\alpha}\right)\left[f_{\alpha}^{\otimes m}(\lambda x^{\alpha}))\right] = m f_{\alpha}^{\otimes m-1}(\lambda x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[f_{\alpha}(\lambda x^{\alpha})]. \tag{30}$$

Proposition 3.1.8: *If $0 < \alpha \leq 1$, μ is a complex number, and g_{α} is a non-zero fractional function,*

$$\left({}_0D_x^{\alpha}\right)\left[g_{\alpha}^{\otimes -1}(\mu x^{\alpha}))\right] = -\left({}_0D_x^{\alpha}\right)[g_{\alpha}(\mu x^{\alpha})] \otimes g_{\alpha}^{\otimes -2}(\mu x^{\alpha}). \tag{31}$$

Proof Since $\left({}_0D_x^{\alpha}\right)\left[(g_{\alpha} \otimes g_{\alpha}^{\otimes -1})(\mu x^{\alpha}))\right] = \left({}_0D_x^{\alpha}\right)[1] = 0$, it follows from Theorem 3.1.5 that

$$\left({}_0D_x^{\alpha}\right)[g_{\alpha}(\mu x^{\alpha})] \otimes g_{\alpha}^{\otimes -1}(\mu x^{\alpha}) + g_{\alpha}(\mu x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)\left[g_{\alpha}^{\otimes -1}(\mu x^{\alpha})\right] = 0.$$

And hence,

$$\left({}_0D_x^{\alpha}\right)\left[g_{\alpha}^{\otimes -1}(\mu x^{\alpha}))\right] = -\left({}_0D_x^{\alpha}\right)[g_{\alpha}(\mu x^{\alpha})] \otimes g_{\alpha}^{\otimes -2}(\mu x^{\alpha}).$$ q.e.d.

Remark 3.1.9: Replacing f_{α} by $f_{\alpha}^{\otimes -1}$ in Eq. (30), and by Eq. (31), we obtain that Corollary 3.1.7 holds for any integer m.

Theorem 3.1.10 (quotient rule for fractional derivatives): *Assume that $0 < \alpha \leq 1$, λ, μ are complex numbers, and f_{α}, g_{α} are fractional functions, $g_{\alpha} \neq 0$, then*

$$\left({}_0D_x^{\alpha}\right)\left[f_{\alpha}(\lambda x^{\alpha}) \otimes g_{\alpha}^{\otimes -1}(\mu x^{\alpha})\right] = g_{\alpha}^{\otimes -2}(\mu x^{\alpha}) \otimes \left(\left({}_0D_x^{\alpha}\right)[f_{\alpha}(\lambda x^{\alpha})] \otimes g_{\alpha}(\mu x^{\alpha}) - f_{\alpha}(\lambda x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[g_{\alpha}(\mu x^{\alpha})]\right).$$

(32)

Proof Using Theorem 3.1.5 and Proposition 3.1.8 yields

$$\left({}_0D_x^{\alpha}\right)\left[f_{\alpha}(\lambda x^{\alpha}) \otimes g_{\alpha}^{\otimes -1}(\mu x^{\alpha})\right]$$

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$$= \left({}_0D_x^{\alpha}\right)[f_\alpha(\lambda x^\alpha)] \otimes g_\alpha^{\otimes -1}(\mu x^\alpha) + f_\alpha(\lambda x^\alpha) \otimes \left({}_0D_x^{\alpha}\right)[g_\alpha^{\otimes -1}(\mu x^\alpha)]$$

$$= \left({}_0D_x^{\alpha}\right)[f_\alpha(\lambda x^\alpha)] \otimes g_\alpha^{\otimes -1}(\mu x^\alpha) + f_\alpha(\lambda x^\alpha) \otimes -\left({}_0D_x^{\alpha}\right)[g_\alpha(\mu x^\alpha)] \otimes g_\alpha^{\otimes -2}(\mu x^\alpha)$$

$$= g_\alpha^{\otimes -2}(\mu x^\alpha) \otimes \left(\left({}_0D_x^{\alpha}\right)[f_\alpha(\lambda x^\alpha)] \otimes g_\alpha(\mu x^\alpha) - f_\alpha(\lambda x^\alpha) \otimes \left({}_0D_x^{\alpha}\right)[g_\alpha(\mu x^\alpha)]\right).$$ q.e.d.

Example 3.1.11: $\left({}_0D_x^{3/5}\right)\left[\frac{2}{\Gamma(9/5+1)}x^{9/5} \otimes \left(1+\frac{4}{\Gamma(3/5+1)}x^{3/5}\right)^{\otimes -1}\right]$

$$= \left(1+\frac{4}{\Gamma(3/5+1)}x^{3/5}\right)^{\otimes -2} \otimes \left(\frac{2}{\Gamma(6/5+1)}x^{6/5} \otimes \left(1+\frac{4}{\Gamma(3/5+1)}x^{3/5}\right) - \frac{2}{\Gamma(9/5+1)}x^{9/5} \otimes 4\right)$$

$$= \left(1+\frac{4}{\Gamma(3/5+1)}x^{3/5}\right)^{\otimes -2} \otimes \left(\frac{2}{\Gamma(6/5+1)}x^{6/5} + \frac{16}{\Gamma(9/5+1)}x^{9/5}\right). \tag{33}$$

Theorem 3.1.12 (Leibniz rule for fractional derivatives): *Let f_α, g_α be fractional functions, λ, μ be complex numbers, and n be a positive integer, then*

$$\left({}_0D_x^{\alpha}\right)^n[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu x^\alpha)] = \sum_{q=0}^{n}\binom{n}{q}\left({}_0D_x^{\alpha}\right)^{n-q}[f_\alpha(\lambda x^\alpha)] \otimes \left({}_0D_x^{\alpha}\right)^{q}[g_\alpha(\mu x^\alpha)]. \tag{34}$$

Proof By product rule for fractional derivatives and induction, we can easily obtain the desired result.

q.e.d.

Example 3.1.13: $\left({}_0D_x^{1/3}\right)^3\left[\frac{1}{\Gamma(5/3+1)}x^{5/3} \otimes E_{1/3}\left(2x^{1/3}\right)\right]$

$$= \sum_{q=0}^{3}\binom{3}{q}\left({}_0D_x^{1/3}\right)^{3-q}\left[\frac{1}{\Gamma(5/3+1)}x^{5/3}\right] \otimes \left({}_0D_x^{1/3}\right)^{q}\left[E_{1/3}\left(2x^{1/3}\right)\right]$$

$$= \left(\frac{8}{\Gamma(5/3+1)}x^{5/3} + \frac{4}{\Gamma(4/3+1)}x^{4/3} + \frac{2}{\Gamma(3/3+1)}x^{3/3}\frac{1}{\Gamma(2/3+1)}x^{2/3}\right) \otimes E_{1/3}\left(2x^{1/3}\right). \tag{35}$$

Theorem 3.1.14 (chain rule for fractional derivatives): *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k p_k(\mu x^\alpha)$. Let $f_{\otimes\alpha}(g_\alpha(\mu x^\alpha)) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}$ and $f'_{\otimes\alpha}(g_\alpha(\mu x^\alpha)) = \sum_{k=1}^{\infty} a_k k\left(g_\alpha(\mu x^\alpha)\right)^{\otimes(k-1)}$, then*

$$\left({}_0D_x^{\alpha}\right)[f_{\otimes\alpha}(g_\alpha(\mu x^\alpha))] = f'_{\otimes\alpha}(g_\alpha(\mu x^\alpha)) \otimes \left({}_0D_x^{\alpha}\right)[g_\alpha(\mu x^\alpha)]. \tag{36}$$

Proof $\left({}_0D_x^{\alpha}\right)[f_{\otimes\alpha}(g_\alpha(\mu x^\alpha))]$

$$= \left({}_0D_x^{\alpha}\right)\left[\sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}\right]$$

$$= \sum_{k=0}^{\infty} a_k \left({}_0D_x^{\alpha}\right)\left[\left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}\right]$$

$$= \sum_{k=1}^{\infty} a_k k\left(g_\alpha(\mu x^\alpha)\right)^{\otimes k-1} \otimes \left({}_0D_x^{\alpha}\right)[g_\alpha(\mu x^\alpha)]$$

$$= f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)].$$ q.e.d.

Example 3.1.15: Let $exp_{\otimes\alpha}\left(sin_\alpha(2x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{\left(sin_\alpha(2x^\alpha)\right)^{\otimes k}}{k!}$, then

$$\left({}_0D_x^\alpha\right)\left[exp_{\otimes\alpha}\left(sin_\alpha(2x^\alpha)\right)\right] = 2 \cdot exp_{\otimes\alpha}\left(sin_\alpha(2x^\alpha)\right) \otimes cos_\alpha(2x^\alpha). \tag{37}$$

The following is the generalization of chain rule for fractional derivatives.

Theorem 3.1.16 (Faà di Bruno formula for fractional derivatives): *If the assumptions are the same as Theorem 3.1.12 and n is a positive integer, then*

$$\left({}_0D_x^\alpha\right)^n\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\right]$$

$$= \sum \frac{n!}{b_1!b_2!\cdots b_n!} f_{\otimes\alpha}^{(k)}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left(\frac{\left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]}{1!}\right)^{\otimes b_1} \otimes \left(\frac{\left({}_0D_x^\alpha\right)^2[g_\alpha(\mu x^\alpha)]}{2!}\right)^{\otimes b_2} \otimes \cdots \otimes \left(\frac{\left({}_0D_x^\alpha\right)^n[g_\alpha(\mu x^\alpha)]}{n!}\right)^{\otimes b_n}, \tag{38}$$

where the sum is taken over all solutions in non-negative integers of $b_1 + 2b_2 + \cdots + nb_n = n$, and $k = b_1 + b_2 + \cdots + b_n$.

Example 3.1.17: $$\left({}_0D_x^\alpha\right)^3\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\right]$$

$$= f'''_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left(\left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]\right)^{\otimes 3}$$

$$+3f''_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)] \otimes \left({}_0D_x^\alpha\right)^2[g_\alpha(\mu x^\alpha)]$$

$$+f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)^3[g_\alpha(\mu x^\alpha)]. \tag{39}$$

3.2 Fractional derivatives of elementary fractional functions

In the following, we make use of the fractional differential properties proved above to find the fractional derivatives of fractional trigonometric functions and hyperbolic functions.

Theorem 3.2.1: *Let $0 < \alpha \le 1$, and b be a real number, then*

$$\left({}_0D_x^\alpha\right)[sin_\alpha(bx^\alpha)] = b \cdot cos_\alpha(bx^\alpha), \tag{40}$$

$$\left({}_0D_x^\alpha\right)[cos_\alpha(bx^\alpha)] = -b \cdot sin_\alpha(bx^\alpha), \tag{41}$$

$$\left({}_0D_x^\alpha\right)\left[tan_\alpha^{\otimes}(bx^\alpha)\right] = b \cdot sec_\alpha^{\otimes 2}(bx^\alpha), \tag{42}$$

$$\left({}_0D_x^\alpha\right)\left[cot_\alpha^{\otimes}(bx^\alpha)\right] = -b \cdot csc_\alpha^{\otimes 2}(bx^\alpha), \tag{43}$$

$$\left({}_0D_x^{\alpha}\right)\left[sec_{\alpha}^{\otimes}(bx^{\alpha})\right] = b \cdot sec_{\alpha}^{\otimes}(bx^{\alpha}) \otimes tan_{\alpha}^{\otimes}(bx^{\alpha}), \tag{44}$$

$$\left({}_0D_x^{\alpha}\right)\left[csc_{\alpha}^{\otimes}(bx^{\alpha})\right] = -b \cdot csc_{\alpha}^{\otimes}(bx^{\alpha}) \otimes cot_{\alpha}^{\otimes}(bx^{\alpha}). \tag{45}$$

Proof

$$\left({}_0D_x^{\alpha}\right)[sin_{\alpha}(bx^{\alpha})] = \left({}_0D_x^{\alpha}\right)\left[\frac{E_{\alpha}(ibx^{\alpha})-E_{\alpha}(-ibx^{\alpha})}{2i}\right]$$

$$= \frac{ibE_{\alpha}(ibx^{\alpha})+ibE_{\alpha}(-ibx^{\alpha})}{2i}$$

$$= \frac{bE_{\alpha}(ibx^{\alpha})+bE_{\alpha}(-ibx^{\alpha})}{2}$$

$$= b \cdot cos_{\alpha}(bx^{\alpha}).$$

$$\left({}_0D_x^{\alpha}\right)[cos_{\alpha}(bx^{\alpha})] = \left({}_0D_x^{\alpha}\right)\left[\frac{E_{\alpha}(ibx^{\alpha})+E_{\alpha}(-ibx^{\alpha})}{2}\right]$$

$$= \frac{ibE_{\alpha}(ibx^{\alpha})-ibE_{\alpha}(-ibx^{\alpha})}{2}$$

$$= \frac{-bE_{\alpha}(ibx^{\alpha})+bE_{\alpha}(-ibx^{\alpha})}{2i}$$

$$= -b \cdot sin_{\alpha}(bx^{\alpha}).$$

$$\left({}_0D_x^{\alpha}\right)\left[tan_{\alpha}^{\otimes}(bx^{\alpha})\right]$$

$$= \left({}_0D_x^{\alpha}\right)\left[sin_{\alpha}(bx^{\alpha}) \otimes cos_{\alpha}^{\otimes -1}(bx^{\alpha})\right]$$

$$= cos_{\alpha}^{\otimes -2}(bx^{\alpha}) \otimes \left(\left({}_0D_x^{\alpha}\right)[sin_{\alpha}(bx^{\alpha})] \otimes cos_{\alpha}(bx^{\alpha}) - sin_{\alpha}(bx^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[cos_{\alpha}(bx^{\alpha})]\right)$$

$$= cos_{\alpha}^{\otimes -2}(bx^{\alpha}) \otimes \left(b \cdot cos_{\alpha}^{2}(bx^{\alpha}) + b \cdot sin_{\alpha}^{2}(bx^{\alpha})\right)$$

$$= b \cdot cos_{\alpha}^{\otimes -2}(bx^{\alpha})$$

$$= b \cdot sec_{\alpha}^{\otimes 2}(bx^{\alpha}).$$

$$\left({}_0D_x^{\alpha}\right)\left[cot_{\alpha}^{\otimes}(bx^{\alpha})\right]$$

$$= \left({}_0D_x^{\alpha}\right)\left[tan_{\alpha}^{\otimes -1}(bx^{\alpha})\right]$$

$$= -\left({}_0D_x^{\alpha}\right)\left[tan_{\alpha}^{\otimes}(bx^{\alpha})\right] \otimes tan_{\alpha}^{\otimes -2}(bx^{\alpha})$$

$$= b \cdot sec_{\alpha}^{\otimes 2}(bx^{\alpha}) \otimes tan_{\alpha}^{\otimes -2}(bx^{\alpha})$$

$$= -b \cdot csc_{\alpha}^{\otimes 2}(bx^{\alpha}).$$

$$\left({}_0D_x^{\alpha}\right)\left[sec_{\alpha}^{\otimes}(bx^{\alpha})\right]$$

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Vol. 7, Issue 5, pp: (1-14), Month: September - October 2020, Available at: www.noveltyjournals.com

$$=\left({}_0D_x^\alpha\right)\left[cos_\alpha^{\otimes -1}(bx^\alpha)\right]$$

$$=-\left({}_0D_x^\alpha\right)[cos_\alpha(bx^\alpha)]\otimes cos_\alpha^{\otimes -2}(bx^\alpha)$$

$$=b\cdot sin_\alpha(bx^\alpha)\otimes cos_\alpha^{\otimes -2}(bx^\alpha)$$

$$=b\cdot sec_\alpha^{\otimes}(bx^\alpha)\otimes tan_\alpha^{\otimes}(bx^\alpha).$$

$$\left({}_0D_x^\alpha\right)\left[csc_\alpha^{\otimes}(bx^\alpha)\right]$$

$$=\left({}_0D_x^\alpha\right)\left[sin_\alpha^{\otimes -1}(bx^\alpha)\right]$$

$$=-\left({}_0D_x^\alpha\right)[sin_\alpha(bx^\alpha)]\otimes sin_\alpha^{\otimes -2}(bx^\alpha)$$

$$=-b\cdot cos_\alpha(bx^\alpha)\otimes sin_\alpha^{\otimes -2}(bx^\alpha)$$

$$=-b\cdot csc_\alpha^{\otimes}(bx^\alpha)\otimes cot_\alpha^{\otimes}(bx^\alpha).$$ q.e.d.

Using the same approach, we can find the fractional derivatives of fractional hyperbolic functions.

Theorem 3.2.2: $$\left({}_0D_x^\alpha\right)[sinh_\alpha(bx^\alpha)]=b\cdot cosh_\alpha(bx^\alpha),$$ (46)

$$\left({}_0D_x^\alpha\right)[cosh_\alpha(bx^\alpha)]=b\cdot sinh_\alpha(bx^\alpha),\tag{47}$$

$$\left({}_0D_x^\alpha\right)\left[tanh_\alpha^{\otimes}(bx^\alpha)\right]=b\cdot sech_\alpha^{\otimes 2}(bx^\alpha),\tag{48}$$

$$\left({}_0D_x^\alpha\right)\left[coth_\alpha^{\otimes}(bx^\alpha)\right]=-b\cdot csch_\alpha^{\otimes 2}(bx^\alpha),\tag{49}$$

$$\left({}_0D_x^\alpha\right)\left[sech_\alpha^{\otimes}(bx^\alpha)\right]=-b\cdot sech_\alpha^{\otimes}(bx^\alpha)\otimes tanh_\alpha^{\otimes}(bx^\alpha),\tag{50}$$

$$\left({}_0D_x^\alpha\right)\left[csch_\alpha^{\otimes}(bx^\alpha)\right]=-b\cdot csch_\alpha^{\otimes}(bx^\alpha)\otimes coth_\alpha^{\otimes}(bx^\alpha).\tag{51}$$

3.3 Application to fractional differential equations

At first, the formal Laurent series of fractional differential operator and its properties are introduced below.

Definition 3.3.1: Assume that $0<\alpha\leq 1$, λ, c_n are complex numbers for all integers n, and z is a complex variable. $h(z)=\sum_{n=-\infty}^{\infty}c_nz^n$ is called a formal Laurent series if $c_n=0$ for all but finitely many negative indices n. And $h\left({}_aD_x^\alpha\right)=\sum_{n=-\infty}^{\infty}c_n\left({}_aD_x^\alpha\right)^n$ is called a formal Laurent series of α-order fractional differential operator.

Proposition 3.3.2: *If* $g(z)=\sum_{k=-\infty}^{\infty}d_kz^k$ *is a formal Laurent series, then*

$$g_{*\alpha}\left(E_\alpha(\lambda x^\alpha)\right)=\sum_{k=-\infty}^{\infty}d_k\left(E_\alpha(\lambda x^\alpha)\right)^{\otimes k}=\sum_{k=-\infty}^{\infty}d_k\,E_\alpha(\lambda kx^\alpha),\tag{52}$$

International Journal of Novel Research in Interdisciplinary Studies
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Theorem 3.3.3: *If* $0 < \alpha \leq 1$, *and* $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, $g(z) = \sum_{k=-\infty}^{\infty} d_k z^k$ *are formal Laurent series, then*

$$\left(f\left({}_0D_x^\alpha\right)\right)\left[g_{\otimes\alpha}\left(E_\alpha(\lambda x^\alpha)\right)\right] = \sum_{k=-\infty}^{\infty} f(\lambda k)\cdot d_k E_\alpha(\lambda k x^\alpha). \tag{53}$$

Proof

$$\left(f\left({}_0D_x^\alpha\right)\right)\left[g_{\otimes\alpha}\left(E_\alpha(\lambda x^\alpha)\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n \left({}_0D_x^\alpha\right)^n \left[\sum_{k=-\infty}^{\infty} d_k\, E_\alpha(\lambda k x^\alpha)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_n\, d_k \left({}_0D_x^\alpha\right)^n \left[E_\alpha(\lambda k x^\alpha)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_n\, d_k (\lambda k)^n E_\alpha(\lambda k x^\alpha)$$

$$= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_n\, d_k (\lambda k)^n E_\alpha(\lambda k x^\alpha)$$

$$= \sum_{k=-\infty}^{\infty} f(\lambda k)\cdot d_k E_\alpha(\lambda k x^\alpha)\,.$$

q.e.d.

The following is a unified form of the particular solution of non-homogeneous linear FDE with constant coefficients.

Theorem 3.3.4 ([23])**:** *If* $0 < \alpha \leq 1$, n *is a positive integer,* $a_0, a_1, \cdots, a_n$ *are real constants, and* $a_n \neq 0$. *The non-homogeneous linear FDE with constant coefficients*

$$\left(a_n\left({}_0D_x^\alpha\right)^n + a_{n-1}\left({}_0D_x^\alpha\right)^{n-1} + \cdots + a_1\left({}_0D_x^\alpha\right) + a_0\right)[y(x^\alpha)] = g_\alpha(x^\alpha) \tag{54}$$

has the particular solution

$$y_p(x^\alpha) = \left(\frac{1}{a_n\left({}_0D_x^\alpha\right)^n + a_{n-1}\left({}_0D_x^\alpha\right)^{n-1} + \cdots + a_1\left({}_0D_x^\alpha\right) + a_0}\right)[g_\alpha(x^\alpha)]. \tag{55}$$

Next, we use the formal Laurent series of fractional differential operator to find the particular solution of non-homogeneous linear FDE with constant coefficients.

Theorem 3.3.5: *If the assumptions are the same as Theorem 3.3.3,* b *is a real number, and let* $g(z) = \sum_{k=-\infty}^{\infty} d_k z^k$ *be a formal Laurent series. Then the non-homogeneous linear FDE with constant coefficients*

$$\left(a_n\left({}_0D_x^\alpha\right)^n + a_{n-1}\left({}_0D_x^\alpha\right)^{n-1} + \cdots + a_1\,{}_0D_x^\alpha + a_0\right)[y(x^\alpha)] = g_{\otimes\alpha}\left(E_\alpha(bx^\alpha)\right) \tag{56}$$

has the particular solution

$$y_p(x^\alpha) = \sum_{k=-\infty}^{\infty} \frac{d_k}{a_n b^n k^n + a_{n-1} b^{n-1} k^{n-1} + \cdots + a_1 bk + a_0}\cdot E_\alpha(bkx^\alpha), \tag{57}$$

if $a_n b^n k^n + a_{n-1} b^{n-1} k^{n-1} + \cdots + a_1 bk + a_0 \neq 0$ *for all* k *with* $d_k \neq 0$.

Proof Since $f(z) = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \sum_{m=-p}^{\infty} c_n z^n$ is a formal Laurent series, it follows that the particular solution of Eq. (56) is

$$y_p(x^\alpha)=\left(f\left({}_aD_x^\alpha\right)\right)\left[g_{\otimes\alpha}\left(E_\alpha(bx^\alpha)\right)\right]$$

$$=\sum_{k=-\infty}^{\infty} f(bk)\cdot d_k E_\alpha(bkx^\alpha)$$

$$=\sum_{k=-\infty}^{\infty}\frac{d_k}{a_nb^nk^n+a_{n-1}b^{n-1}k^{n-1}+\cdots+a_1bk+a_0}\cdot E_\alpha(bkx^\alpha).\qquad \text{q.e.d.}$$

Example 3.3.6: Let $x>0$. Consider the non-homogeneous linear FDE with constant coefficients

$$\left(2\left({}_0D_x^{1/2}\right)^2+5\left({}_0D_x^{1/2}\right)+1\right)\left[y\left(x^{1/2}\right)\right]=E_{1/2}\left(12x^{1/2}\right)\otimes\left(1-E_{1/2}\left(-3x^{1/2}\right)\right)^{\otimes-1}. \tag{58}$$

Since

$$E_{1/2}\left(12x^{1/2}\right)\otimes\left(1-E_{1/2}\left(-3x^{1/2}\right)\right)^{\otimes-1}$$

$$=\left(E_{1/2}\left(-3x^{1/2}\right)\right)^{\otimes-4}\otimes\sum_{k=0}^{\infty}\left(E_{1/2}\left(-3x^{1/2}\right)\right)^{\otimes k}$$

$$=\sum_{k=-4}^{\infty}\left(E_{1/2}\left(-3x^{1/2}\right)\right)^{\otimes k}.$$

It follows from Theorem 3.3.5 that the particular solution of Eq. (58) is

$$y_p\left(x^{1/2}\right)=\sum_{k=-4}^{\infty}\frac{1}{18k^2-15k+1}\cdot E_{1/2}\left(-3kx^{1/2}\right). \tag{59}$$

IV. CONCLUSION

The basic fractional differential properties of fractional functions are correct under the new multiplication such that we can obtain the fractional derivatives of some elementary fractional functions. In fact, these fundamental properties of fractional functions are the generalizations of the ones of classical elementary functions. On the other hand, we use the formal Laurent series of fractional differential operator method to obtain the particular solution of non-homogeneous linear FDE with constant coefficients, regarding Jumarie type of modified R-L fractional derivative. At the same time, our approach is also the generalization of the method to find the particular solution of classical non-homogeneous linear differential equations with constant coefficients.

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